Semigroups, a tool to develop Harmonic Analysis for general laplacians

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Historical sources

Pioners in the study of Harmonic Analysis for laplacians.

B. Muckenhoupt, Poisson integrals for Hermite and Laguerre expansions, *Trans. Amer. Math. Soc.* **139** (1969), 231-242.

B. Muckenhoupt, Hermite conjugate expansions, *Trans. Amer. Math. Soc.* **139** (1969), 244-260.

B. Muckenhoupt and E. M. Stein Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17-92.

Some ideas in Muckenhoupt's

The obvious Poisson integral for a function $f(y)$ with Hermite expansion $\sum a_n H_n(y)$ is the function $g(r, y)$ with Hermite expansion $\sum r^n a_n H_n(y)$, $0 \leq r < 1$.

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application of the general theorem in §2. An alternate Poisson integral, $f(x, y)$, is also mentioned. If $f(y)$ has the Hermite expansion given above, $f(x, y)$ is the function which for fixed $x > 0$ has the expansion $\sum a_n \exp[-(2n)^{1/2}x]H_n(y)$. The theorems proved for g are immediately applicable to this since there is a simple relation between it and g. Like the ordinary Poisson integral, $f(x, y)$ satisfies a second order elliptic differential equation. In fact, $f_{11}(x, y) + f_{22}(x, y) - 2yf_2(x, y) = 0$. This makes $f(x, y)$ a more reasonable Poisson integral and makes it possible to

define conjugate functions for Hermite expansions. These conjugate functions will be treated in another paper.

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It was shown in [2] that

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \exp(y^2)\frac{\partial}{\partial y}\left(\exp(-y^2)\frac{\partial f(x, y)}{\partial y}\right) = 0. \tag{1.1}$$

Similarly, it will be shown here that

$$\frac{\partial^2 \tilde{f}(x, y)}{\partial x^2} + \frac{\partial}{\partial y}\left[\exp(y^2)\frac{\partial}{\partial y}(\exp(-y^2)\tilde{f}(x, y))\right] = 0 \tag{1.2}$$

and that the analogues of the Cauchy-Riemann equations

$$\frac{\partial f(x, y)}{\partial x} = \exp(y^2)\frac{\partial}{\partial y}(\exp(-y^2)\tilde{f}(x, y)) \tag{1.3}$$

Careful reading of Muckenhoupt's words

Hermite polynomials.

- Rodrigues' $H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \quad n = 0, 1, 3 \ldots$
- Orthogonal with respect to $d\gamma(x) = e^{-x^2} dx$.
- Ornstein-Uhlenbeck, $L_x = -\Delta + 2x\nabla = -\partial_x^2 + 2x\partial_x, \quad L_x H_n = 2nH_n$.

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1.- Poisson sums.

$$f(x) = \sum_n a_n H_n(x), \quad g(r,x) = \sum_n r^n a_n H_n(x) = \int_{\mathbb{R}} \sum_n r^n \frac{H_n(x)H_n(z)}{\|H_n\|^2} f(z)dz.$$

$$\sum_n r^n \frac{H_n(x)H_n(z)}{\|H_n\|^2} = P_r(x,y) = \frac{1}{\pi^{1/2}(1-r^2)^{1/2}} exp\left(\frac{-r^2x^2 + 2rxz - r^2z^2}{1-r^2}\right).$$

P_r parallel properties to the classical Poisson sums. Convergence a.e. for $L^p(d\gamma), 1 \le p < \infty$. L^p-convergence $1 < p < \infty$, ...

2.- Harmonic functions. Cauchy-Riemann equations

He consider $f(x,t) = \sum_n e^{-(2n)^{1/2}t} a_n H_n(x) = \int_{\mathbb{R}} \mathcal{P}_t(x,y) f(y) d\gamma(y)$.

It seems natural to get the "harmonicity"

$$(\partial_t^2 - L_x) f = 0, \quad \text{(observe that } \partial_t^2 e^{-(2n)^{1/2}t} H_n = L_x e^{-(2n)^{1/2}t} H_n).$$

Careful reading of Muckenhoupt's words (cont)

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Factorization $L = (\partial_x)^* \partial_x = (-\partial_x + 2x)\partial_x$. $((\partial_x)^*$ with respect to $d\gamma)$.

Cauchy-Riemann

$$\tilde{f}(t,x) = \sum_n (2n)^{1/2} e^{-(2n)^{1/2}t} a_n H_{n-1}(x).$$

$$\partial_t f(t,x) = (-\partial_x + 2x)\tilde{f}(t,x), \quad \partial_x f(t,x) = \partial_t \tilde{f}(t,x).$$

Careful reading of Muckenhoupt's words (cont)

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(Remember C-R equations $\partial_t u(x,t) = -\partial_x v(x,t)$, $\partial_x u(x,t) = \partial_t v(x,t)$).
u, v harmonic functions.

Carefull reading of Muckenhoupt's words (last)

$$f \sim \sum_n a_n H_n.$$

$$g(r(t),x) = \sum_n r^n a_n H_n \underbrace{=}_{(r=e^{-2t})} \sum_n e^{-t2n} a_n H_n. \quad \text{Poisson.}$$

$$f(t,x) = \sum_n e^{-t(2n)^{1/2}} a_n H_n. \quad \text{conjugate.}$$

Formally

$$(\partial_t + L) g(r(t),x) = 0 \quad \text{“heat ” equation. !!Poisson is misleading!!.}$$

$$(\partial_t^2 - L) f(t,x) = 0 \quad \text{Poisson equation, Harmonic function.}$$

Review of classical case. Fourier Series – Poisson sums.

$$f(\theta) = \sum_k a_k e^{ik\theta}. \qquad -\Delta_\theta f = -\partial_\theta^2 f = \textstyle\sum_k |k|^2 a_k e^{ik\theta}$$

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$$\underbrace{=}_{(z=re^{i\theta})} 1 + \sum_{k>0} a_k\, z^k + \sum_{k>0} r^k a_{-k}\, \bar{z}^k = U(z)$$

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Harmonic.

$$\left(r^2\partial_r^2 + r\partial_r + \Delta_\theta\right) P_r f(\theta) = \left(r^2\partial_r^2 + r\partial_r + \partial_\theta^2\right) P_r f(\theta) = 0 \quad \Longleftrightarrow \quad \partial_z\, \partial_{\bar{z}}\, U = 0$$

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$(r\partial_r)^* = -r\partial_r$ with respect to $d\mu(r) = \frac{dr}{r}$ en $[0,1]$.

$(\partial_\theta)^* = -\partial_\theta$ with respect to $d\theta$.

Decomposition:

$$\Delta_\theta = -(\partial_\theta)^*(\partial_\theta) \quad \text{and} \quad r^2\partial_r^2 + r\partial_r + \Delta_\theta = -(r\partial_r)^*(r\partial_r) + \Delta_\theta.$$

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$$\begin{aligned} Q_r f(\theta) &= -i \sum_{k\neq 0} \text{sign}\, k\, r^{|k|} a_k\, e^{ik\theta} \\ &= -i \sum_{k>0} r^k a_k\, e^{ik\theta} + i \sum_{k>0} r^k a_{-k}\, e^{-ik\theta} \\ &\underbrace{=}_{(z=re^{i\theta})} -i \sum_{k>0} a_k\, z^k + i \sum_{k>0} r^k a_{-k}\, \bar{z}^k = V(z) \end{aligned}$$

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$$\left(r^2\partial_r^2 + r\partial_r + \Delta_\theta\right) Q_r f(\theta) = \left(-(r\partial_r)^*(r\partial_r) + \Delta_\theta\right) Q_r f(\theta) = 0. \quad \Longleftrightarrow \quad \partial_z \, \partial_{\bar{z}} \, V = 0.$$

Cauchy–Riemann equations (Fourier series)

$$P_r f(\theta) = \sum_k r^{|k|} a_k \, e^{ik\theta} = 1 + \sum_{k>0} a_k \, z^k + \sum_{k>0} r^k a_{-k} \, \bar{z}^k = U(z)$$

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Cauchy-Riemann (Fourier series)

$$\partial_\theta \, (P_r f)(\theta) = -r\partial_r (Q_r f)(\theta)$$

$$r\partial_r \, (P_r f)(\theta) = \partial_\theta \, (Q_r f)(\theta) \quad \Big(\text{i.e. } (r\partial_r)^* \, (P_r f)(\theta) = (\partial_\theta)^* \, (Q_r f)(\theta)\Big)$$

Review of the classical case. On $\mathbb{R}$.

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} d\xi. \qquad -\Delta f(x) = -\partial_x^2 f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) |\xi|^2 e^{ix\xi} d\xi.$$

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$$P_t f(x) = \int_{\mathbb{R}} e^{-t|\xi|} \widehat{f}(\xi)\, e^{i\xi x} d\xi$$
$$\underbrace{=}_{(z=t-ix)} \int_0^\infty e^{-z\xi} \widehat{f}(\xi) d\xi + \int_0^\infty e^{-\bar{z}\xi} \widehat{f}(-\xi) d\xi = U(z)$$

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Harmonic.

$$\left(\partial_t^2 + \Delta_x\right) P_t f(x) = \left(\partial_t^2 + \partial_x^2\right) P_t f(x) = 0 \quad \Longleftrightarrow \quad \partial_z \, \partial_{\bar{z}} \, U = 0$$

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$(\partial_x)^* = -\partial_x$ with respect to dx
$(\partial_t)^* = -\partial_t$ with respect to dt

Descomposition.

$$\Delta_x = -(\partial_x)^*(\partial_x) \quad \text{and} \quad \partial_t^2 + \Delta_x = -\Big[(\partial_t)^*(\partial_t) + (\partial_x)^*(\partial_x)\Big]$$

Conjugate harmonic function (line)

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$$\left(\partial_t^2 + \Delta_x\right) Q_t f(x) = 0 \quad \Longleftrightarrow \quad \partial_z \, \partial_{\bar{z}} \, V = 0$$

Cauchy–Riemann equations (line)

$$P_t f(x) = \int_{\mathbb{R}} e^{-t|\xi|}\widehat{f}(\xi)\, e^{i\xi\, x} d\xi = \int_0^{\infty} e^{-z\xi}\widehat{f}(\xi) d\xi + \int_0^{\infty} e^{-\bar{z}\xi}\widehat{f}(-\xi) d\xi = U(z)$$

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Cauchy–Riemann (line)

$$\partial_x P_t f(x) = \partial_t Q_t f(x) \quad \Big(\text{i.e. } (\partial_x)^* P_t f(x) = (\partial_t)^* Q_t f(x)\Big).$$

$$\partial_t P_t f(x) = -\partial_x Q_t f(x).$$

How to understand the apparently incongruence of the operators:

$$f \sim \sum a_n Hn \longrightarrow g(r(t), x) = \sum_n r^n H_n = \sum_n e^{-t2n} a_n H_n,$$

and

$$f \sim \sum a_n Hn \longrightarrow f(t, x) = \sum_n e^{-t(2n)^{1/2}} a_n H_n,$$

defined by Muckenhoupt, with their classical parallels?

Diffusion semigroups

E. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley theory*, Princeton, 1970.

Definition

Symmetric diffusion semi-groups

$(\mathcal{M}, d\mu)$measure space . $\{T_t\}_{t>0} : L^2 \to L^2$:

- $T_{t_1+t_2} = T_{t_1} T_{t_2}$. $T_0 = Id$. $\lim_{t\to 0} T_t f = f$ in L^2.
- $\|T_t f\|_p \leq \|f\|_p$, $(1 \leq p \leq \infty)$. Contraction.
- T_t selfadjoint in L^2.
- $T_t f \geq 0$ si $f \geq 0$. Positivity.
- $T_t 1 = 1$. Markov.

First example of semigroup. Classical heat equation

The example of diffusion semigroup in $L^2(\mathbb{R})$:

$$T_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy.$$

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For good functions f we have:

$$\partial_t (T_t f(x)) = \partial_x^2 (T_t f(x)) = \Delta_x (T_t f(x)).$$

In symbols

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Useful remark

$$\widehat{T_t f}(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi).$$

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$$T_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy.$$

For good functions f we have:

$$\partial_t(T_t f(x)) = \partial_x^2(T_t f(x)) = \Delta_x(T_t f(x)).$$

In symbols

$$T_t f(x) = e^{t\Delta} f(x) \quad \text{“heat semigroup”}.$$

Useful remark

$$\widehat{T_t f}(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi).$$

Remember

$$\widehat{P_t f}(\xi) = e^{-t\sqrt{|\xi|^2}} \widehat{f}(\xi).$$

Second example of diffusion semigroup. Orthogonal systems.

Ilustration. L with eigenfunctions $\{\phi_k\}_k$ and eigenvalues $\{\lambda_k\}_k$

$$e^{-tL}\phi_k(x) = e^{-t\lambda_k}\phi_k(x), \qquad e^{-tL}\Big(\sum_k c_k\phi_k\Big)(x) = \sum_k e^{-t\lambda_k}c_k\phi_k(x).$$

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Hermite polynomials case. $L = -\partial_x^2 + 2x\partial_x, \quad LH_n = 2nH_n$. Then

$$e^{-tL}\Big(\sum_n a_n H_n\Big)(x) = \sum_n e^{-t2n} a_n H_n(x).$$

(remember Muckenhoupt).

Poisson semigroup. Fourier series case

Formula for Gamma function:

$$e^{-t\sqrt{\lambda}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} e^{-s\lambda} ds, \quad \lambda > 0.$$

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Fourier series case.

$e^{-t(-\Delta)}\left(\sum_k a_k e^{-ik\theta}\right) = \sum_k e^{-t|k|^2} a_k e^{-ik\theta}$. We can define

$$\begin{aligned} e^{-t\sqrt{-\Delta}}\left(\sum_k a_k e^{-ik\theta}\right) &= \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} e^{-s(-\Delta)}\left(\sum_k a_k e^{-ik\theta}\right) ds \\ &= \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} \left(\sum_k e^{-t|k|^2} a_k e^{-ik\theta}\right) ds \\ &= \sum_k e^{-t|k|} a_k e^{-ik\theta} \end{aligned}$$

Poisson semigroup. Hermite case

Hermite polynomials. Heat, $e^{-tL}\Big(\sum_n a_n H_n\Big)(x) = \sum_n e^{-t2n} a_n H_n(x).$
Poisson:

$$\begin{aligned} e^{-t\sqrt{L}}\Big(\sum_n a_n H_n(x)\Big) &= \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}}\, e^{-s(-\Delta)}\Big(\sum_n e^{-t2n} a_n H_n(x)\Big) ds \\ &= \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}}\Big(\sum_n e^{-t2n} a_n H_n(x)\Big) ds \\ &= \sum_n e^{-t(2n)^{1/2}} a_n H_n(x). \end{aligned}$$

In general given a positive operator L. (Bochner subordination formula).

$$e^{-t\sqrt{L}} = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}}\, e^{-sL} ds, \quad \lambda > 0. \qquad (\partial_{tt}^2 - L)e^{-t\sqrt{L}} = 0$$

Determination of kernels

If we know e^{-tL} then we know $e^{-t\sqrt{L}}$.

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$$e^{-t\sqrt{(-\Delta)}}(x,y) = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}}\, e^{-s(-\Delta)}(x,y)\, ds$$

$$= \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\frac{1}{(4\pi s)^{n/2}} e^{-\frac{|x-y|^2}{4s}}\, ds$$

$$\underbrace{=}_{(\frac{t^2+|x-y|^2}{4s}=u)} \frac{t}{\pi^{(n+1)/2}}\int_0^\infty \frac{u^{(n+1)/2}}{(t^2+|x-y|^2)^{(n+1)/2}}\, e^{-u}\, du$$

$$= \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}\frac{t}{(t^2+|x-y|^2)^{(n+1)/2}}.$$

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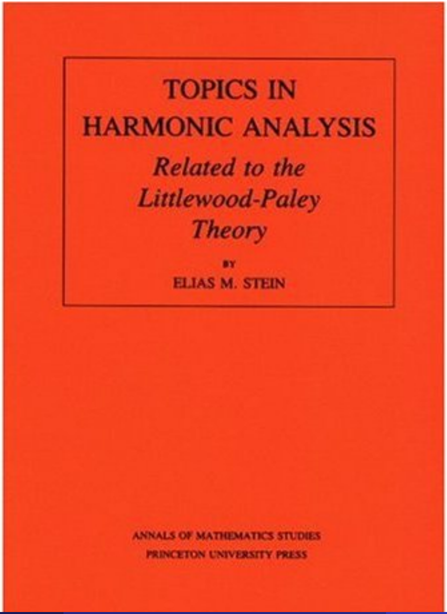
&= \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} \frac{1}{(4\pi s)^{n/2}} e^{-\frac{|x-y|^2}{4s}}\, ds \\
&\underbrace{=}_{\left(\frac{t^2+|x-y|^2}{4s}=u\right)} \frac{t}{\pi^{(n+1)/2}} \int_0^\infty \frac{u^{(n+1)/2}}{(t^2+|x-y|^2)^{(n+1)/2}}\, e^{-u}\, du \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \frac{t}{(t^2+|x-y|^2)^{(n+1)/2}}.
\end{aligned}
$$

Conclusion. If we know the kernel of the heat semigroup, we know the kernel of the Poisson semigroup.

Stein knew everything!.

Stein knew everything!. Look into the red book. **¡1970!**



Stein proposed some "heuristics" ...

We assume that G is a non-compact, connected, Lie group. We let $X_1, X_2, \ldots, X_n$ be a basis for the (left-invariant) Lie algebra, considered as first-order differential operators on G. We set

$$\Delta^+ = \Sigma\ a_{ij} X_i X_j$$

where $\{a_{ij}\}$ is *any* real symmetric positive definite matrix. (More specific choices of the $\{a_{ij}\}$ will be made later.) Our first object is to consider the heat-diffusion semigroup $T_+^t = e^{t\Delta^+}$.

The definition of the Riesz transforms can be given symbolically as

$$R_i(f) = \bar{X}_i(-\Delta)^{-1/2} f\ .$$

“Fractional integrals”

Gamma function formula: $\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-\lambda t} \frac{dt}{t}, \quad \lambda, \alpha > 0.$

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Given ϕ_k, eigenfunction of L with eigenvalue λ_k, we have

$$L^{-\alpha}\phi_k = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-tL}\phi_k \frac{dt}{t} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-t\lambda_k}\phi_k \frac{dt}{t} = \frac{1}{\lambda_k^\alpha}\phi_k.$$

In general for positive L,

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In general for positive L,

$$L^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha \, e^{-tL} f \, \frac{dt}{t}$$

$$L^{-1/2} f = (\sqrt{L})^{-1} f = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{1/2} \, e^{-tL} f \, \frac{dt}{t} = \frac{1}{\Gamma(1)} \int_0^\infty e^{-t\sqrt{L}} f \, dt.$$

Kernel of fractional integrals (on $\mathbb{R}^n$).

$$\begin{aligned}
(-\Delta_x)^{-\alpha/2} f(x) &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2}\, e^{-t(-\Delta_x)} f(x)\, \frac{dt}{t} \\
&= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} f(y) dy t^{\alpha/2} \frac{dt}{t} \\
&= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} t^{\alpha/2} \frac{dt}{t} f(y) dy \\
&= \frac{1}{(4\pi)^{n/2}} \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} \int_0^\infty \frac{4^{(n-\alpha)/2}}{|x-y|^{n-\alpha}} u^{(n-\alpha)/2} \frac{du}{u} f(y) dy \\
&= \frac{1}{\pi^{n/2} 4^{\alpha/2} \Gamma(\alpha/2)} \Gamma\left(\frac{n-\alpha}{2}\right) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) dy \\
&= c_{n,\alpha} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) dy.
\end{aligned}$$

We don't use Fourier transform!!!

“Riesz Transforms”. Boundedness in L^2

On $\mathbb{R}^n$,

$$R_i f = \partial_{x_i}(-\Delta)^{-1/2} f, \quad i = 1, \dots, n$$

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Eigenvalue case (one dimension), assume $L = \partial^* \partial$, $\quad \psi_k = \partial (L)^{-1/2} \phi_k$

$$\int \psi_k \psi_\ell d\mu = \int \left(\partial (L)^{-1/2} \phi_k\right)\left(\partial (L)^{-1/2} \phi_\ell\right) \, d\mu$$

$$= \int \left(\partial^* \partial (L)^{-1/2} \phi_k\right)\left((L)^{-1/2} \phi_\ell\right) d\mu = \int \left(L (L)^{-1/2} \phi_k\right)\left((L)^{-1/2} \phi_\ell\right) \, d\mu$$

$$= \int \left((L)^{1/2} \phi_k\right)\left((L)^{-1/2} \phi_\ell\right) \, d\mu = \lambda_k^{1/2} \, \lambda_\ell^{-1/2} \int \phi_k \, \phi_\ell \, d\mu.$$

This gives boundedness in $L^2(d\mu)$ of $Rf = \partial L^{-1/2}$.

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$$= \int \left((L)^{1/2}\phi_k\right)\left((L)^{-1/2}\phi_\ell\right) d\mu = \lambda_k^{1/2}\,\lambda_\ell^{-1/2} \int \phi_k\,\phi_\ell\, d\mu.$$

This gives boundedness in $L^2(d\mu)$ of $Rf = \partial L^{-1/2}$.

For a positive operator L we have:

$$\langle \partial_x (L)^{-1/2} f, \partial_x (L)^{-1/2} f\rangle = \langle \partial_x^* \partial_x (L)^{-1/2} f, (L)^{-1/2} f\rangle$$
$$= \langle (L)^{1/2} f, (L)^{-1/2} f\rangle = \langle f, f\rangle$$

Boundedness in L^p of Riesz transforms.

General procedure in Harmonic Analysis. Once we know the L^2-boundedness, the kernel is used to get L^p boundedness.

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\begin{aligned}
\partial_x L^{-1/2} f(x) &= \partial_x \int_0^\infty e^{-tL} f(x)\, t^{1/2}\, \frac{dt}{t} = \int_0^\infty \partial_x e^{-tL} f(x)\, t^{1/2}\, \frac{dt}{t} \\
&= \int_0^\infty \partial_x \int_{\mathbb{R}^n} e^{-tL}(x,y) f(y)\, dy\, t^{1/2}\, \frac{dt}{t} \\
&= \int_{\mathbb{R}^n} \left(\int_0^\infty \partial_x\, e^{-tL}(x,y)\, t^{1/2}\, \frac{dt}{t} \right) f(y)\, dy \\
&= \int_{\mathbb{R}^n} K(x,y) f(y)\, dy.
\end{aligned}
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Exercise. For good enough functions

$$\partial_{x_i}(-\Delta)^{-1/2} f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} P.V. \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^{n+1}} f(y)dy$$

Without using Fourier Transform !!.

Conjugate functions

$L = \partial_x^* \partial_x$. With Poisson semigroup $u(x,t) = e^{-t\sqrt{L}} f(x)$.
Satisfies the Poisson equation $\partial_{tt}^2 u - Lu = 0$.

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Observe that

$$\partial_x^* v(x,t) = \partial_x^* \partial_x (L)^{-1/2} e^{-t\sqrt{L}} f = (L)^{1/2} e^{-t\sqrt{L}} f = -\partial_t e^{-t\sqrt{L}} f = -\partial_t u(x,t)$$

and

$$\partial_t v(x,t) = -\partial_x (L)^{-1/2} \sqrt{L} e^{-t\sqrt{L}} f = -\partial_x u(x,t)$$

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Then

$$\partial_{tt}^2 v(x,t) = -\partial_x \partial_t u = \partial_x \partial_x^* v(x,t) = \tilde{L} v(x,t)$$

Words of Muckenhoupt are clear now. Define $\tilde{f} = \partial_x L^{-1/2} e^{-t\sqrt{L}}$.

Application to discrete laplacian.

O.Ciaurri, A. Gillespie, L.Roncal, J.L.T. , J.L. Varona (J.Analyse Math.)

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$$u_t(n,t) = \Delta_{\rm d} u(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t), \quad n \in \mathbb{Z},$$

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$I_k(t)$ Bessel function of imaginary argument.

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Formal series $e^{-t(-\Delta_{\mathrm{d}})} f(n) = W_t f(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m)$.

The function $u(n,t) = W_t f(n)$ is the solution to the heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t), \\ u(n,0) = f(n), \end{cases}$$

where u is the unknown function and the sequence $f = \{f(n)\}_{n\in\mathbb{Z}}$ is the initial datum at time $t = 0$.

$$e^{-t(-\Delta_{\mathrm{d}})}f(n) = W_t f(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m).$$

Proposition

Let $f \in \ell^\infty$. The family $\{W_t\}_{t\geq 0}$ satisfies

(i) $W_0 f = f$.

(ii) $W_{t_1} W_{t_2} f = W_{t_1+t_2} f$.

(iii) *If* $f \in \ell^2$ *then* $W_t f \in \ell^2$ *and* $\lim_{t\to 0} W_t f = f$ *in* ℓ^2.

(iv) *(Contraction property)* $\|W_t f\|_{\ell^p} \leq \|f\|_{\ell^p}$ *for* $1 \leq p \leq +\infty$.

(v) *(Positivity preserving)* $W_t f \geq 0$ *if* $f \geq 0$, $f \in \ell^2$.

(vi) *(Markovian property)* $W_t 1 = 1$.

some formulas

$$I_k(t) = i^{-k} J_k(it) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+k+1)} \left(\frac{t}{2}\right)^{2m+k}.$$

Since k is an integer (and $1/\Gamma(n)$ is taken to equal zero if $n = 0, -1, -2, \ldots$), the function I_k is defined in the whole real line (even in the whole complex plane, where I_k is an entire function).

$$I_{-k}(t) = I_k(t).$$

$$I_r(t_1 + t_2) = \sum_{k \in \mathbb{Z}} I_k(t_1) I_{r-k}(t_2) \quad \text{for} \quad r \in \mathbb{Z}. \quad \text{Neumann's identity.}$$

$I_k(t) \geq 0$ for every $k \in \mathbb{Z}$ and $t \geq 0$, and $\sum_{k \in \mathbb{Z}} e^{-2t} I_k(2t) = 1$.

$$I_k(t) = C e^t t^{-1/2} + R_k(t), \quad |R_k(t)| \leq C_k e^t t^{-3/2}, \quad \text{for } t \to \infty.$$

$$\frac{\partial}{\partial t} I_k(t) = \frac{1}{2}(I_{k+1}(t) + I_{k-1}(t)),$$

and from this it follows immediately

$$\frac{\partial}{\partial t}(e^{-2t} I_k(2t)) = e^{-2t}(I_{k+1}(2t) - 2I_k(2t) + I_{k-1}(2t)).$$

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Schläfli's integral

$$I_\nu(z) = \frac{z^\nu}{\sqrt{\pi}\, 2^\nu \Gamma(\nu + 1/2)} \int_{-1}^{1} e^{-zs}(1 - s^2)^{\nu - 1/2}\, ds, \quad |\arg z| < \pi, \quad \nu > -\frac{1}{2}.$$

$$I_{\nu+1}(z) - I_\nu(z) = -\frac{z^\nu}{\sqrt{\pi}\, 2^\nu \Gamma(\nu + 1/2)} \int_{-1}^{1} e^{-zs}(1 + s)(1 - s^2)^{\nu - 1/2}\, ds.$$

$$I_{\nu+2}(z) - 2I_{\nu+1}(z) + I_\nu(z) = \frac{z^\nu}{\sqrt{\pi}\, 2^\nu \Gamma(\nu + 1/2)} \times \Big(\frac{2}{z} \int_{-1}^{1} e^{-zs} s(1 - s^2)^{\nu - 1/2}\, ds + \int_{-1}^{1} e^{-zs}(1 + s)^2(1 - s^2)^{\nu - 1/2}\, ds \Big).$$

Proposition

Let $f \in \ell^\infty$. Then,

$$u(n,t) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m), \quad t>0, \quad n \in \mathbb{Z},$$

is a solution of the heat equation.

$$\partial_t u(n,t) - \Delta_{\mathrm{d}} u(n,t) = 0.$$

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Remark

$$P_t f(n) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} W_{t^2/(4u)} f(n)\, du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4v)}}{\sqrt{v}} W_v f(n) \frac{dv}{v}$$

satisfies the "Laplace" equation

$$\partial_{tt}^2 P_t f(n) + \Delta_{\mathrm{d}} P_t f(n) = 0.$$

Discrete Riesz transforms

Consider the "first" order difference operators

$$Df(n) = f(n+1) - f(n) \quad \text{and} \quad \widetilde{D}f(n) = f(n) - f(n-1),$$

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"Riesz transforms":

$$\mathcal{R} = D(-\Delta_{\mathrm{d}})^{-1/2} \quad \text{and} \quad \widetilde{\mathcal{R}} = \widetilde{D}(-\Delta_{\mathrm{d}})^{-1/2}. \tag{1}$$

Operator $D(-\Delta_{\mathrm{d}})^{-1/2}$ is given by the multiplier $-ie^{-i\theta/2}$ and also described as the convolution with the kernel $\frac{1}{\pi(n+\frac{1}{2})}$.
$\tilde{D}(-\Delta_{\mathrm{d}})^{-1/2}$ is given by the multiplier $-ie^{i\theta/2}$ and also described as the convolution with the kernel $\frac{1}{\pi(n-\frac{1}{2})}$.
This gives roundedness in ℓ^2. (Also ℓ^p)

Discrete conjugate functions

$$Q_t f = \mathcal{R} P_t f \quad \text{and} \quad \widetilde{Q}_t f = \widetilde{\mathcal{R}} P_t f.$$

Proposition

Let Q_t and $\widetilde{Q}_t$ as above and f be a compactly supported function.

(i) *The operators Q_t, $\widetilde{Q}_t$ and P_t satisfy the Cauchy–Riemann type equations*

$$\begin{cases} \partial_t(Q_t f) = -D(P_t f), \\ \widetilde{D}(Q_t f) = \partial_t(P_t f); \end{cases} \qquad \begin{cases} \partial_t(\widetilde{Q}_t f) = -\widetilde{D}(P_t f), \\ D(\widetilde{Q}_t f) = \partial_t(P_t f). \end{cases}$$

Moreover, $\partial_{tt}^2 Q_t f(n) + \Delta_{\mathrm{d}} Q_t f(n) = 0$ and $\partial_{tt}^2 \widetilde{Q}_t f(n) + \Delta_{\mathrm{d}} \widetilde{Q}_t f(n) = 0$.

(ii) *We have*

$$\lim_{t\to 0} Q_t f(n) = \mathcal{R} f(n) \quad \text{and} \quad \lim_{t\to 0} \widetilde{Q}_t f(n) = \widetilde{\mathcal{R}} f(n),$$

for $n \in \mathbb{Z}$.

Other applications

In PDE's.
-Analysis of the positive powers L^α. "Fractional laplacians" for general "L" With P. Stinga. Comm. Partial Differential Equations. (2010).

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-Analysis of lateral operators. Work in progress with A. Bernardis, F. J.Martín-Reyes, P.Stinga, J.L.T.
- Numerical Analysis of discretized operators. Work in progress with O.Ciaurri, L. Roncal, P.Stinga, J.L.T., J.L. Varona.

¡Muchas gracias por su atención!