Semigroups, a tool to develop Harmonic Analysis for general laplacians

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Historical sources

Pioners in the study of Harmonic Analysis for laplacians.


Some ideas in Muckenhoupt’s

The obvious Poisson integral for a function \( f(y) \) with Hermite expansion \( \sum a_n H_n(y) \) is the function \( g(r, y) \) with Hermite expansion \( \sum r^n a_n H_n(y), \quad 0 \leq r < 1. \)
The obvious Poisson integral for a function $f(y)$ with Hermite expansion $\sum a_n H_n(y)$ is the function $g(r, y)$ with Hermite expansion $\sum r^n a_n H_n(y)$, $0 \leq r < 1$. An alternate Poisson integral, $f(x, y)$, is also mentioned. If $f(y)$ has the Hermite expansion given above, $f(x, y)$ is the function which for fixed $x > 0$ has the expansion $\sum a_n \exp \left[-(2n)^{1/2}x\right] H_n(y)$. The theorems proved for $g$ are immediately applicable to this since there is a simple relation between it and $g$. Like the ordinary Poisson integral, $f(x, y)$ satisfies a second order elliptic differential equation. In fact, $f_{11}(x, y) + f_{22}(x, y) - 2yf_2(x, y) = 0$. This makes $f(x, y)$ a more reasonable Poisson integral and makes it possible to define conjugate functions for Hermite expansions. These conjugate functions will be treated in another paper.
Some ideas in Muckenhoupt’s

The obvious Poisson integral for a function \( f(y) \) with Hermite expansion
\[
\sum a_n H_n(y)
\]
is the function \( g(r, y) \) with Hermite expansion \( \sum r^na_n H_n(y) \), \( 0 \leq r < 1 \).

application of the general theorem in §2. An alternate Poisson integral, \( f(x, y) \), is also mentioned. If \( f(y) \) has the Hermite expansion given above, \( f(x, y) \) is the function which for fixed \( x > 0 \) has the expansion \( \sum a_n \exp \left[ -(2n)^{1/2} x \right] H_n(y) \). The theorems proved for \( g \) are immediately applicable to this since there is a simple relation between it and \( g \). Like the ordinary Poisson integral, \( f(x, y) \) satisfies a second order elliptic differential equation. In fact, \( f_{11}(x, y) + f_{22}(x, y) - 2y f_2(x, y) = 0 \).

This makes \( f(x, y) \) a more reasonable Poisson integral and makes it possible to define conjugate functions for Hermite expansions. These conjugate functions will be treated in another paper.

It was shown in [2] that
\[
(1.1) \quad \frac{\partial^2 f(x, y)}{\partial x^2} + \exp (y^2) \frac{\partial}{\partial y} \left( \exp (-y^2) \frac{\partial f(x, y)}{\partial y} \right) = 0.
\]

Similarly, it will be shown here that
\[
(1.2) \quad \frac{\partial^2 \tilde{f}(x, y)}{\partial x^2} + \frac{\partial}{\partial y} \left[ \exp (y^2) \frac{\partial}{\partial y} (\exp (-y^2) \tilde{f}(x, y)) \right] = 0
\]
and that the analogues of the Cauchy-Riemann equations
\[
(1.3) \quad \frac{\partial f(x, y)}{\partial x} = \exp (y^2) \frac{\partial}{\partial y} (\exp (-y^2) f(x, y))
\]
Hermite polynomials.

- Rodrigues’ \( H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n} \), \( n = 0, 1, 3 \ldots \)
- Orthogonal with respect to \( d\gamma(x) = e^{-x^2} dx \).
- Ornstein-Uhlenbeck, \( L_x = -\Delta + 2x\nabla = -\partial_x^2 + 2x\partial_x \), \( L_x H_n = 2nH_n \).
Careful reading of Muckenhoupt’s words

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1.- Poisson sums.

\[ f(x) = \sum_n a_n H_n(x), \quad g(r, x) = \sum_n r^n a_n H_n(x) = \int_{\mathbb{R}} \sum_n r^n \frac{H_n(x)H_n(z)}{\|H_n\|^2} f(z) dz. \]

\[ \sum_n r^n \frac{H_n(x)H_n(z)}{\|H_n\|^2} = P_r(x, y) = \frac{1}{\pi^{1/2}(1 - r^2)^{1/2}} \exp\left( \frac{-r^2 x^2 + 2rxz - r^2 z^2}{1 - r^2} \right). \]

\( P_r \) parallel properties to the classical Poisson sums. Convergence a.e. for \( L^p(d\gamma), 1 \leq p < \infty. \) \( L^p \)–convergence \( 1 < p < \infty, \ldots \)
Careful reading of Muckenhoupt’s words (cont)

2.- Harmonic functions. Cauchy-Riemann equations

He consider \( f(x, t) = \sum e^{-(2n)^{1/2} t} a_n H_n(x) = \int_\mathbb{R} \mathcal{P}_t(x, y) f(y) d\gamma(y). \)

It seems natural to get the “harmonicity”

\[(\partial_t^2 - L_x) f = 0, \quad \text{(observe that } \partial_t^2 e^{-(2n)^{1/2} t} H_n = L_x e^{-(2n)^{1/2} t} H_n). \]
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Factorization $L = (\partial_x)^* \partial_x = (-\partial_x + 2x) \partial_x. \quad ((\partial_x)^* \text{ with respect to } d\gamma)$. Cauchy-Riemann

$$\tilde{f}(t, x) = \sum_n (2n)^{1/2} e^{-(2n)^{1/2} t} a_n H_{n-1}(x).$$

$$\partial_t f(t, x) = (-\partial_x + 2x) \tilde{f}(t, x), \quad \partial_x f(t, x) = \partial_t \tilde{f}(t, x).$$
2.- Harmonic functions. Cauchy-Riemann equations

He consider $f(x, t) = \sum_n e^{-(2n)^{1/2}t} a_n H_n(x) = \int_\mathbb{R} \mathcal{P}_t(x, y)f(y)d\gamma(y)$.

It seems natural to get the “harmonicity”

$$\left(\partial_t^2 - L_x\right)f = 0,$$

(observe that $\partial_t^2 e^{-(2n)^{1/2}t} H_n = L_x e^{-(2n)^{1/2}t} H_n$).

Factorization $L = (\partial_x)^* \partial_x = (-\partial_x + 2x)\partial_x$. (($\partial_x)^*$ with respect to $d\gamma$).

Cauchy-Riemann

$$\tilde{f}(t, x) = \sum_n (2n)^{1/2} e^{-(2n)^{1/2}t} a_n H_{n-1}(x).$$

$$\partial_t f(t, x) = (-\partial_x + 2x)\tilde{f}(t, x), \quad \partial_x f(t, x) = \partial_t \tilde{f}(t, x).$$

(Remember C-R equations $\partial_t u(x, t) = -\partial_x v(x, t), \quad \partial_x u(x, t) = \partial_t v(x, t)$).

$u, v$ harmonic functions.
Carefull reading of Muckenhoupt’s words (last)

\[ f \sim \sum_n a_n H_n. \]

\[ g(r(t), x) = \sum_n r^n a_n H_n = \sum_n e^{-t^{2n}} a_n H_n. \quad \text{Poisson.} \]

\[ f(t, x) = \sum_n e^{-t(2n)^{1/2}} a_n H_n. \quad \text{conjugate.} \]

Formally

\[ (\partial_t + L)g(r(t), x) = 0 \quad \text{“heat ” equation. !!!Poisson is misleading!!}. \]

\[ (\partial_t^2 - L)f(t, x) = 0 \quad \text{Poisson equation, Harmonic function.} \]

\[ f(\theta) = \sum_k a_k e^{ik\theta}. \quad -\Delta_\theta f = -\partial^2_\theta f = \sum_k |k|^2 a_k e^{ik\theta} \]

\[ f(\theta) = \sum_k a_k e^{ik\theta}. \quad -\Delta_\theta f = -\partial_\theta^2 f = \sum_k |k|^2 a_k e^{ik\theta} \]

\[ P_r f(\theta) = \sum_k r^{|k|} a_k e^{ik\theta} = \sum_k r^{\sqrt{|k|^2}} a_k e^{ik\theta} = 1 + \sum_{k>0} r^k a_k e^{ik\theta} + \sum_{k>0} r^k a_{-k} e^{-ik\theta} \]

\[ \underbrace{1 + \sum_{k>0} a_k z^k}_{(z=re^{i\theta})} + \sum_{k>0} r^k a_{-k} \bar{z}^k = U(z) \]

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**Harmonic.**

\[ (r^2 \partial_r^2 + r \partial_r + \Delta_\theta) P_r f(\theta) = (r^2 \partial_r^2 + r \partial_r + \partial^2_\theta) P_r f(\theta) = 0 \iff \partial_z \partial_{\bar{z}} U = 0 \]

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\[ = 1 + \sum_{k>0} a_k z^k + \sum_{k>0} r^k a_{-k} \bar{z}^k = U(z) \quad (z=re^{i\theta}) \]

Harmonic.

\[ (r^2 \partial^2_r + r \partial_r + \Delta\theta) P_r f(\theta) = (r^2 \partial^2_r + r \partial_r + \partial^2_\theta) P_r f(\theta) = 0 \quad \iff \quad \partial_z \partial_{\bar{z}} U = 0 \]

\[(r \partial_r)^* = -r \partial_r \text{ with respect to } d\mu(r) = \frac{dr}{r} \text{ en } [0,1]. \]

\[(\partial_\theta)^* = -\partial_\theta \text{ with respect to } d\theta. \]

Decomposition:

\[ \Delta\theta = -(\partial_\theta)^*(\partial_\theta) \quad \text{and} \quad r^2 \partial^2_r + r \partial_r + \Delta\theta = -(r \partial_r)^*(r \partial_r) + \Delta\theta. \]

\[ f(\theta) = \sum_{k} a_k e^{ik\theta} \]

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\[ Q_r f(\theta) = -i \sum_{k \neq 0} \text{sign } k \ r|k| \ a_k \ e^{ik\theta} \]

\[ = -i \sum_{k > 0} r^k a_k \ e^{ik\theta} + i \sum_{k > 0} r^k a_{-k} \ e^{-ik\theta} \]

\[ = -i \sum_{k > 0} a_k z^k + i \sum_{k > 0} r^k a_{-k} \overline{z}^k = V(z) \]

\( (z = re^{i\theta}) \)}

Fourier series.

\[ f(\theta) = \sum_k a_k e^{ik\theta} \]

\[ Q_r f(\theta) = -i \sum_{k \neq 0} \text{sign } k \ r^{|k|} a_k e^{ik\theta} \]

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\[ = -i \sum_{k>0} a_k z^k + i \sum_{k>0} r^k a_{-k} \bar{z}^k = V(z) \]

Harmonic.

\[ (r^2 \partial_r^2 + r \partial_r + \Delta_\theta) \ Q_r f(\theta) = \left( -(r \partial_r)^* (r \partial_r) + \Delta_\theta \right) Q_r f(\theta) = 0. \quad \iff \quad \partial_z \partial_{\bar{z}} V = 0. \]
Cauchy–Riemann equations (Fourier series)

\[ P_r f(\theta) = \sum_k r^{|k|} a_k e^{ik\theta} = 1 + \sum_{k>0} a_k z^k + \sum_{k>0} r^k a_{-k} \bar{z}^k = U(z) \]

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\[ F(z) = U(z) + iV(z) = 1 + 2 \sum_{k>0} a_k z^k \quad \text{is holomorphic: } \partial_z F = 0 \]
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Cauchy-Riemann (Fourier series)

\[ \partial_\theta (P_r f)(\theta) = -r \partial_r (Q_r f)(\theta) \]

\[ r \partial_r (P_r f)(\theta) = \partial_\theta (Q_r f)(\theta) \quad \text{(i.e. } (r \partial_r)^* (P_r f)(\theta) = (\partial_\theta)^* (Q_r f)(\theta)) \]
Review of the classical case. On $\mathbb{R}$.

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi.$$  

$$-\Delta f(x) = -\partial_x^2 f(x) = \int_{\mathbb{R}} \hat{f}(\xi)|\xi|^2 e^{ix\xi} d\xi.$$
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$$P_t f(x) = \int_{\mathbb{R}} e^{-t|\xi|} \hat{f}(\xi) e^{ix}\hat{f}(\xi) d\xi$$

$$= \int_{0}^{\infty} e^{-z\xi} \hat{f}(\xi) d\xi + \int_{0}^{\infty} e^{-z\xi} \hat{f}(-\xi) d\xi = U(z)$$
Review of the classical case. On $\mathbb{R}$.

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$$= \underbrace{\int_0^\infty e^{-z\xi} \hat{f}(\xi) \, d\xi}_{(z=t-ix)} + \int_0^\infty e^{-z\xi} \hat{f}(-\xi) \, d\xi = U(z)$$

Harmonic.

$$\left( \partial_t^2 + \Delta_x \right) P_t f(x) = \left( \partial_t^2 + \partial_x^2 \right) P_t f(x) = 0 \iff \partial_z \partial_{\overline{z}} U = 0$$
Review of the classical case. On $\mathbb{R}$.

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi.$$  
$$-\Delta f(x) = -\partial_x^2 f(x) = \int_{\mathbb{R}} \hat{f}(\xi)|\xi|^2 e^{ix\xi} d\xi.$$  

$$P_t f(x) = \int_{\mathbb{R}} e^{-t|\xi|^2} \hat{f}(\xi) e^{ix\xi} d\xi = \int_0^\infty e^{-z\xi} \hat{f}(\xi) d\xi + \int_0^\infty e^{-z\xi} \hat{f}(-\xi) d\xi = U(z)$$

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$$(\partial_t^2 + \Delta_x) P_t f(x) = (\partial_t^2 + \partial_x^2) P_t f(x) = 0 \iff \partial_z \partial_{\bar{z}} U = 0$$

($\partial_x^*) = -\partial_x$ with respect to $dx$
($\partial_t^*) = -\partial_t$ with respect to $dt$

Descomposition.

$$\Delta_x = -(\partial_x^*)(\partial_x) \quad \text{and} \quad \partial_t^2 + \Delta_x = -\left[ (\partial_t^*)(\partial_t) + (\partial_x^*)(\partial_x) \right]$$
Conjugate harmonic function (line)

\[ f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi \]
Conjugate harmonic function (line)

\[ f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi \]

\[ Q_t f(x) = -i \int_{\mathbb{R}} \text{sign } \xi \ e^{-t|\xi|} \hat{f}(\xi) e^{i\xi x} d\xi \]

\[ = -i \int_{0}^{\infty} e^{-z\xi} \hat{f}(\xi) d\xi + i \int_{0}^{\infty} e^{-\bar{z}\xi} \hat{f}(-\xi) d\xi = V(z) \]
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\[ = -i \int_0^\infty e^{-z\xi} \hat{f}(\xi) d\xi + i \int_0^\infty e^{-\bar{z}\xi} \hat{f}(-\xi) d\xi = V(z) \]

Harmonic.

\[ (\partial_t^2 + \Delta_x) \, Q_t f(x) = 0 \iff \partial_z \partial_{\bar{z}} \, V = 0 \]
Cauchy–Riemann equations (line)

\[ P_t f(x) = \int_{\mathbb{R}} e^{-t|\xi|} \hat{f}(\xi) e^{ix} d\xi = \int_{0}^{\infty} e^{-z\xi} \hat{f}(\xi) d\xi + \int_{0}^{\infty} e^{-\bar{z}\xi} \hat{f}(-\xi) d\xi = U(z) \]

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\[ F(z) = U(z) + iV(z) = 2 \int_{0}^{\infty} e^{-z\xi} \hat{f}(\xi) d\xi \quad \text{is holomorphic:} \quad \partial_{\bar{z}} F = 0 \]
Cauchy–Riemann equations

\[ P_t f(x) = \int_{\mathbb{R}} e^{-t|\xi|} \hat{f}(\xi) e^{i\xi \cdot x} d\xi = \int_0^\infty e^{-z\xi} \hat{f}(\xi) d\xi + \int_0^\infty e^{-\bar{z}\xi} \hat{f}(-\xi) d\xi = U(z) \]

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Cauchy–Riemann (line)

\[ \partial_x P_t f(x) = \partial_t Q_t f(x) \quad (\text{i.e. } (\partial_x)^* P_t f(x) = (\partial_t)^* Q_t f(x)). \]

\[ \partial_t P_t f(x) = -\partial_x Q_t f(x). \]
How to understand the apparently incongruence of the operators:

\[ f \sim \sum a_n H_n \longrightarrow g(r(t), x) = \sum r^n H_n = \sum e^{-t^{2n}} a_n H_n, \]

and

\[ f \sim \sum a_n H_n \longrightarrow f(t, x) = \sum e^{-t(2n)^{1/2}} a_n H_n, \]

defined by Muckenhoupt, with their classical parallels?

**Definition**

Symmetric diffusion semi-groups

\((\mathcal{M}, d\mu)\) measure space . \(\{T_t\}_{t>0} : L^2 \rightarrow L^2 :\)

- \(T_{t_1 + t_2} = T_{t_1} T_{t_2}\). \(T_0 = Id. \lim_{t \to 0} T_t f = f \) in \(L^2.\)
- \(\|T_t f\|_p \leq \|f\|_p, \ (1 \leq p \leq \infty). \) Contraction.
- \(T_t\) selfadjoint in \(L^2.\)
- \(T_t f \geq 0 \) si \(f \geq 0. \) Positivity.
- \(T_t 1 = 1. \) Markov.
The example of diffusion semigroup in $L^2(\mathbb{R})$:

$$T_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy.$$
First example of semigroup. Classical heat equation

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$$T_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy.$$ 

For good functions $f$ we have:

$$\partial_t (T_t f(x)) = \partial_x^2 (T_t f(x)) = \Delta_x (T_t f(x)).$$

In symbols

$$T_t f(x) = e^{t\Delta} f(x) \quad \text{“heat semigroup”}.$$
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Useful remark

$$\hat{T_t f}(\xi) = e^{-t|\xi|^2} \hat{f}(\xi).$$
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Useful remark

$$\hat{T}_t f(\xi) = e^{-t|\xi|^2} \hat{f}(\xi).$$

Remember

$$\hat{P}_t f(\xi) = e^{-t\sqrt{|\xi|^2}} \hat{f}(\xi).$$
Second example of diffusion semigroup. Orthogonal systems.

Illustration. $L$ with eigenfunctions $\{\phi_k\}_k$ and eigenvalues $\{\lambda_k\}_k$

\[
e^{-tL} \phi_k(x) = e^{-t\lambda_k} \phi_k(x), \quad e^{-tL} \left( \sum_k c_k \phi_k \right)(x) = \sum_k e^{-t\lambda_k} c_k \phi_k(x).
\]
Second example of diffusion semigroup. Orthogonal systems.

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We have the “heat” equation for \( L \):

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\partial_t (e^{-tL} f(x)) = -Le^{-tL} f(x).
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Fourier series case. $-\Delta e^{ik\theta} = |k|^2 e^{-ik\theta}$.

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\]

---

**Hermite polynomials** case. \( L = -\partial_x^2 + 2x\partial_x \), \( LH_n = 2nH_n \). Then

\[
e^{-tL} \left( \sum_n a_n H_n \right)(x) = \sum_n e^{-t^{2n}} a_n H_n(x).
\]

(remember Muckenhoupt).
Poisson semigroup. Fourier series case

Formula for Gamma function:

\[ e^{-t\sqrt{\lambda}} = \frac{t}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-t^2/4s}}{s^{3/2}} e^{-s\lambda} \, ds, \quad \lambda > 0. \]
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Fourier series case.

\[ e^{-t(-\Delta)} \left( \sum_k a_k e^{-ik\theta} \right) = \sum_k e^{-t|k|^2} a_k e^{-ik\theta}. \]  

We can define

\[ e^{-t\sqrt{-\Delta}} \left( \sum_k a_k e^{-ik\theta} \right) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} e^{-s(-\Delta)} \left( \sum_k a_k e^{-ik\theta} \right) \, ds \]

\[ = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} \left( \sum_k e^{-t|k|^2} a_k e^{-ik\theta} \right) \, ds \]

\[ = \sum_k e^{-t|k|^2} a_k e^{-ik\theta} \]
**Poisson semigroup. Hermite case**

**Hermite polynomials.** Heat, 

\[ e^{-tL} \left( \sum_n a_n H_n \right)(x) = \sum_n e^{-t^2n} a_n H_n(x). \]

Poisson:

\[
e^{-t\sqrt{L}} \left( \sum_n a_n H_n(x) \right) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} e^{-s(-\Delta)} \left( \sum_n e^{-t^2n} a_n H_n(x) \right) ds
\]

\[
= \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} \left( \sum_n e^{-t^2n} a_n H_n(x) \right) ds
\]

\[
= \sum_n e^{-t(2n)^{1/2}} a_n H_n(x).
\]

---

In general given a positive operator \( L \). (Bochner subordination formula).

\[
e^{-t\sqrt{L}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} e^{-sL} ds, \quad \lambda > 0. \quad (\partial_{tt}^2 - L)e^{-t\sqrt{L}} = 0
\]
Determination of kernels

If we know $e^{-tL}$ then we know $e^{-t\sqrt{L}}$. 

Conclusion.
If we know the kernel of the heat semigroup, we know the kernel of the Poisson semigroup.
Determination of kernels

If we know $e^{-tL}$ then we know $e^{-t\sqrt{L}}$.

\[ e^{-t\sqrt{(-\Delta)}}(x, y) = \frac{t}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} e^{-s(-\Delta)}(x, y) \, ds \]

\[ = \frac{t}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} \frac{1}{(4\pi s)^{n/2}} e^{-\frac{|x-y|^2}{4s}} \, ds \]

\[ = \frac{t}{\pi^{(n+1)/2}} \int_{0}^{\infty} \frac{u^{(n+1)/2}}{(t^2 + |x-y|^2)^{(n+1)/2}} e^{-u} \, du \]

\[ = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}}. \]
Conclusion. If we know the kernel of the heat semigroup, we know the kernel of the Poisson semigroup.
Stein knew everything!.
Stein knew everything! Look into the red book. ¡1970!
We assume that $G$ is a non-compact, connected, Lie group. We let $X_1, X_2, \ldots, X_n$ be a basis for the (left-invariant) Lie algebra, considered as first-order differential operators on $G$. We set

$$\Delta^+ = \sum a_{ij} X_i X_j$$

where $\{a_{ij}\}$ is any real symmetric positive definite matrix. (More specific choices of the $\{a_{ij}\}$ will be made later.) Our first object is to consider the heat-diffusion semigroup $T^t_+ = e^{t\Delta^+}$.

The definition of the Riesz transforms can be given symbolically as

$$R_i(f) = \tilde{X}_i (-\Delta)^{-\frac{1}{2}} f.$$
Gamma function formula: $\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha} e^{-\lambda t} \frac{dt}{t}$, $\lambda, \alpha > 0$. 
“Fractional integrals”

Gamma function formula: \( \lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-\lambda t} \frac{dt}{t}, \quad \lambda, \alpha > 0. \)

Given \( \phi_k \), eigenfunction of \( L \) with eigenvalue \( \lambda_k \), we have

\[
L^{-\alpha} \phi_k = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-tL} \phi_k \frac{dt}{t} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-t \lambda_k} \phi_k \frac{dt}{t} = \frac{1}{\lambda_k^\alpha} \phi_k.
\]

In general for positive \( L \),

\[
L^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-tL} f \frac{dt}{t}
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In general for positive \( L \),

\[ L^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-tL} f \frac{dt}{t}. \]

\[ L^{-1/2} f = (\sqrt{L})^{-1} f = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{1/2} e^{-tL} f \frac{dt}{t} = \frac{1}{\Gamma(1)} \int_0^\infty e^{-t\sqrt{L}} f \ dt. \]
Kernel of fractional integrals (on $\mathbb{R}^n$).

\[
(-\Delta_x)^{-\alpha/2}f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2} e^{-t(-\Delta_x)} f(x) \frac{dt}{t} = \frac{1}{\Gamma(\alpha/2)} \int_\mathbb{R}^n \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} f(y) dy t^{\alpha/2} \frac{dt}{t} = \frac{1}{\Gamma(\alpha/2)} \int_\mathbb{R}^n \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} t^{\alpha/2} \frac{dt}{t} f(y) dy = \frac{1}{(4\pi)^{n/2}} \frac{1}{\Gamma(\alpha/2)} \int_\mathbb{R}^n \int_0^\infty \frac{4^{(n-\alpha)/2}}{|x-y|^{n-\alpha}} u^{(n-\alpha)/2} \frac{du}{u} f(y) dy = \frac{1}{\pi^{n/2} 4^{\alpha/2} \Gamma(\alpha/2)} \Gamma\left(\frac{n-\alpha}{2}\right) \int_\mathbb{R}^n \frac{1}{|x-y|^{n-\alpha}} f(y) dy = c_{n,\alpha} \int_\mathbb{R}^n \frac{1}{|x-y|^{n-\alpha}} f(y) dy.
\]

We don’t use Fourier transform!!!
“Riesz Transforms”. Boundedness in $L^2$

On $\mathbb{R}^n$, 

\[ R_i f = \partial x_i (-\Delta)^{-1/2} f, \quad i = 1, \ldots, n \]
"Riesz Transforms". Boundedness in $L^2$

On $\mathbb{R}^n$,

$$R_i f = \partial_{x_i} (-\Delta)^{-1/2} f, \quad i = 1, \ldots, n$$

Eigenvalue case (one dimension), assume $L = \partial^* \partial$,

$$\psi_k = \partial (L)^{-1/2} \phi_k$$

$$\int \psi_k \psi_\ell d\mu = \int \left( \partial (L)^{-1/2} \phi_k \right) \left( \partial (L)^{-1/2} \phi_\ell \right) d\mu$$

$$= \int \left( \partial^* \partial (L)^{-1/2} \phi_k \right) \left( (L)^{-1/2} \phi_\ell \right) d\mu = \int \left( L (L)^{-1/2} \phi_k \right) \left( (L)^{-1/2} \phi_\ell \right) d\mu$$

$$= \int \left( (L)^{1/2} \phi_k \right) \left( (L)^{-1/2} \phi_\ell \right) d\mu = \lambda_k^{1/2} \lambda_\ell^{-1/2} \int \phi_k \phi_\ell d\mu.$$ 

This gives boundedness in $L^2(d\mu)$ of $Rf = \partial L^{-1/2}$. 

( UAM)
"Riesz Transforms". Boundedness in $L^2$

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This gives boundedness in $L^2(d\mu)$ of $Rf = \partial L^{-1/2}$.

For a positive operator $L$ we have:

$$\langle \partial_x (L)^{-1/2} f, \partial_x (L)^{-1/2} f \rangle = \langle \partial^*_x \partial_x (L)^{-1/2} f, (L)^{-1/2} f \rangle$$

$$= \langle (L)^{1/2} f, (L)^{-1/2} f \rangle = \langle f, f \rangle$$
Boundedness in $L^p$ of Riesz transforms.

**General procedure in Harmonic Analysis.** Once we know the $L^2$-boundedness, the kernel is used to get $L^p$ boundedness.

\[
\partial_x \left( -\Delta \right)^{-1/2} f(x) = \Gamma \left( \frac{n+1}{2} \right) \pi^{\frac{n+1}{2}} P.V. \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^{n+1}} f(y) \, dy.
\]

*Exercise.* For good enough functions \[ \partial_x^i \left( -\Delta \right)^{-1/2} f(x) = \Gamma \left( \frac{n+1}{2} \right) \pi^{\frac{n+1}{2}} P.V. \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^{n+1}} f(y) \, dy. \] Without using Fourier Transform!!
Boundedness in $L^p$ of Riesz transforms.

**General procedure in Harmonic Analysis.** Once we know the $L^2$-boundedness, the kernel is used to get $L^p$ boundedness.

$$\partial_x L^{-1/2} f(x) = \partial_x \int_0^\infty e^{-tL} f(x) \frac{t^{1/2}}{t} \, dt = \int_0^\infty \partial_x e^{-tL} f(x) \frac{t^{1/2}}{t} \, dt$$

$$= \int_0^\infty \partial_x \left( \int_{\mathbb{R}^n} e^{-tL} (x, y) f(y) \, dy \right) \frac{t^{1/2}}{t} \, dt$$

$$= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \partial_x e^{-tL} (x, y) \frac{t^{1/2}}{t} \, dt \right) f(y) \, dy$$

$$= \int_{\mathbb{R}^n} K(x, y) f(y) \, dy.$$
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$$= \int_0^\infty \partial_x \int_{\mathbb{R}^n} e^{-tL}(x, y) f(y) \, dy \frac{t^{1/2}}{t} \, dt$$

$$= \int_{\mathbb{R}^n} \left( \int_0^\infty \partial_x e^{-tL}(x, y) \frac{t^{1/2}}{t} \, dt \right) f(y) \, dy$$

$$= \int_{\mathbb{R}^n} K(x, y) f(y) \, dy.$$

Exercise. For good enough functions

$$\partial_{x_i} (-\Delta)^{-1/2} f(x) = \left[ \Gamma \left( \frac{n+1}{2} \right) \right] P.V. \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x - y|^{n+1}} f(y) \, dy$$

Without using Fourier Transform !!.
Conjugate functions

$L = \partial^*_x \partial_x$. With Poisson semigroup $u(x, t) = e^{-t\sqrt{L}}f(x)$.
Satisfies the Poisson equation \( \partial_{tt}u - Lu = 0 \).
Conjugate functions

\[ L = \partial^*_x \partial_x. \]  With Poisson semigroup \( u(x, t) = e^{-t\sqrt{L}}f(x). \)
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Define the “Conjugate operator” \( v(x, t) = \partial_x(L)^{-1/2}e^{-t\sqrt{L}}f(x). \)
Conjugate functions

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Define the “Conjugate operator” \( v(x, t) = \partial_x (L)^{-1/2} e^{-t\sqrt{L}} f(x). \)

Observe that

\[ \partial_x^* v(x, t) = \partial_x^* \partial_x (L)^{-1/2} e^{-t\sqrt{L}} f = (L)^{1/2} e^{-t\sqrt{L}} f = -\partial_t e^{-t\sqrt{L}} f = -\partial_t u(x, t) \]

and

\[ \partial_t v(x, t) = -\partial_x (L)^{-1/2} \sqrt{L} e^{-t\sqrt{L}} f = -\partial_x u(x, t) \]
Conjugate functions

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Then

\[ \partial_{tt}^2 v(x, t) = -\partial_x \partial_t u = \partial_x \partial_x^* v(x, t) = \tilde{L} v(x, t) \]
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Then

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Words of Muckenhoupt are clear now. Define \( \tilde{f} = \partial_x L^{-1/2} e^{-t\sqrt{L}}. \)
Application to discrete laplacian.

O.Ciaurri, A. Gillespie, L.Roncal, J.L.T., J.L. Varona (J.Analyse Math.)

\[ \Delta_d f(n) = f(n+1) - 2f(n) + f(n-1), \quad n \in \mathbb{Z}. \]

The solution of
\[ u_t(n, t) = \Delta_d u(n, t) = u(n+1, t) - 2u(n, t) + u(n-1, t), \quad n \in \mathbb{Z}, \]
\[ u(n, 0) = \delta_{nm} \] for every fixed \( m \in \mathbb{Z} \), is given by
\[ u(n, t) = e^{-2t} \text{I}_n^{-m}(2t). \]

I\( _k \) Bessel function of imaginary argument.

Formal series
\[ e^{-t}(\Delta_d f(n)) = W_t f(n) = \sum_{m \in \mathbb{Z}} e^{-2t} \text{I}_n^{-m}(2t) f(m). \]

The function
\[ u(n, t) = W_t f(n) \]

is the solution to the heat equation
\[ \partial_t u(n, t) = u(n+1, t) - 2u(n, t) + u(n-1, t), \]
where \( u \) is the unknown function and the sequence \( f = \{ f(n) \} \) \( n \in \mathbb{Z} \) is the initial datum at time \( t = 0 \).
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\[ u(n, 0) = \delta_{nm} \text{ for every fixed } m \in \mathbb{Z}, \text{ is given by } u(n, t) = e^{-2t} I_{n-m}(2t). \]
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\[ I_k(t) \quad \text{Bessel function of imaginary argument.} \]

Formal series

\[ e^{-t(-\Delta_d)} f(n) = W_t f(n) = \sum_{m \in \mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m). \]

The function \[ u(n, t) = W_t f(n) \]

is the solution to the heat equation

\[ \begin{cases} 
\frac{\partial}{\partial t} u(n, t) = u(n + 1, t) - 2u(n, t) + u(n - 1, t), \\
u(n, 0) = f(n),
\end{cases} \]

where \( u \) is the unknown function and the sequence \( f = \{ f(n) \}_{n \in \mathbb{Z}} \) is the initial datum at time \( t = 0 \).
\[ e^{-t(-\Delta_d)} f(n) = W_t f(n) = \sum_{m \in \mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m). \]

**Proposition**

Let \( f \in \ell^\infty \). The family \( \{W_t\}_{t \geq 0} \) satisfies

(i) \( W_0 f = f \).
(ii) \( W_{t_1} W_{t_2} f = W_{t_1 + t_2} f \).
(iii) If \( f \in \ell^2 \) then \( W_t f \in \ell^2 \) and \( \lim_{t \to 0} W_t f = f \) in \( \ell^2 \).
(iv) (Contraction property) \( \|W_t f\|_{\ell^p} \leq \|f\|_{\ell^p} \) for \( 1 \leq p \leq +\infty \).
(v) (Positivity preserving) \( W_t f \geq 0 \) if \( f \geq 0 \), \( f \in \ell^2 \).
(vi) (Markovian property) \( W_t 1 = 1 \).
some formulas

\[ I_k(t) = i^{-k} J_k(it) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + k + 1)} \left( \frac{t}{2} \right)^{2m+k}. \]

Since \( k \) is an integer (and \( 1/\Gamma(n) \) is taken to equal zero if \( n = 0, -1, -2, \ldots \)), the function \( I_k \) is defined in the whole real line (even in the whole complex plane, where \( I_k \) is an entire function).

\[ I_{-k}(t) = I_k(t). \]

\[ I_r(t_1 + t_2) = \sum_{k \in \mathbb{Z}} I_k(t_1)I_{r-k}(t_2) \quad \text{for} \quad r \in \mathbb{Z}. \quad \text{Neumann’s identity.} \]

\( I_k(t) \geq 0 \) for every \( k \in \mathbb{Z} \) and \( t \geq 0 \), and \( \sum_{k \in \mathbb{Z}} e^{-2t} I_k(2t) = 1. \)

\[ I_k(t) = Ce^t t^{-1/2} + R_k(t), \quad |R_k(t)| \leq C_k e^t t^{-3/2}, \quad \text{for} \quad t \to \infty. \]
\[
\frac{\partial}{\partial t} I_k(t) = \frac{1}{2} (I_{k+1}(t) + I_{k-1}(t)),
\]
and from this it follows immediately
\[
\frac{\partial}{\partial t} (e^{-2t} I_k(2t)) = e^{-2t} (I_{k+1}(2t) - 2I_k(2t) + I_{k-1}(2t)).
\]
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Schläfli’s integral

\[
I_\nu(z) = \frac{z^\nu}{\sqrt{\pi} 2^\nu \Gamma(\nu + 1/2)} \int_{-1}^{1} e^{-zs} (1 - s^2)^{\nu-1/2} \, ds, \quad |\arg z| < \pi, \quad \nu > -\frac{1}{2}.
\]

\[ I_{\nu+1}(z) - I_{\nu}(z) = -\frac{z^\nu}{\sqrt{\pi} 2^\nu \Gamma(\nu + 1/2)} \int_{-1}^{1} e^{-zs} (1 + s)(1 - s^2)^{\nu-1/2} \, ds. \]

\[ I_{\nu+2}(z) - 2I_{\nu+1}(z) + I_{\nu}(z) = \frac{z^\nu}{\sqrt{\pi} 2^\nu \Gamma(\nu + 1/2)} \times \left( \frac{2}{z} \int_{-1}^{1} e^{-zs} s(1 - s^2)^{\nu-1/2} \, ds + \int_{-1}^{1} e^{-zs} (1 + s)^2(1 - s^2)^{\nu-1/2} \, ds \right). \]
Let \( f \in \ell^\infty \). Then,

\[
\sum_{m \in \mathbb{Z}} e^{-2t} I_{n-m}(2t)f(m), \quad t > 0, \quad n \in \mathbb{Z},
\]
is a solution of the heat equation.

\[
\partial_t u(n, t) - \Delta_d u(n, t) = 0.
\]
Proposition

Let \( f \in \ell^\infty \). Then,

\[
u(n, t) = \sum_{m \in \mathbb{Z}} e^{-2t} I_{n-m}(2t)f(m), \quad t > 0, \quad n \in \mathbb{Z},
\]

is a solution of the heat equation.

\[
\partial_t u(n, t) - \Delta_d u(n, t) = 0.
\]

Remark

\[
P_t f(n) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} W_{t^2/(4u)} f(n) \, du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4\nu)}}{\sqrt{\nu}} W_{\nu} f(n) \, d\nu
\]

satisfies the “Laplace” equation

\[
\partial_{tt} P_t f(n) + \Delta_d P_t f(n) = 0.
\]
Consider the “first” order difference operators

\[ Df(n) = f(n + 1) - f(n) \quad \text{and} \quad \tilde{D}f(n) = f(n) - f(n - 1), \]

that allow factorization of the discrete Laplacian as \( \Delta_d = \tilde{D}D. \)
Consider the “first” order difference operators

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that allow factorization of the discrete Laplacian as \( \Delta_d = \tilde{D}D. \)

“Riesz transforms”:

\[ \mathcal{R} = D(-\Delta_d)^{-1/2} \quad \text{and} \quad \tilde{\mathcal{R}} = \tilde{D}(-\Delta_d)^{-1/2}. \]  

Operator \( D(-\Delta_d)^{-1/2} \) is given by the multiplier \(-ie^{-i\theta}/2\) and also described as the convolution with the kernel \(1/\pi(n+1/2).\)

\( \tilde{D}(-\Delta_d)^{-1/2} \) is given by the multiplier \(-ie^{i\theta}/2\) and also described as the convolution with the kernel \(1/\pi(n-1/2).\)

This gives roundedness in \( \ell^2. \) (Also \( \ell^p \))
Proposition

Let $Q_t$ and $\tilde{Q}_t$ as above and $f$ be a compactly supported function.

(i) The operators $Q_t$, $\tilde{Q}_t$ and $P_t$ satisfy the Cauchy–Riemann type equations

\[
\begin{align*}
\partial_t( Q_t f) &= -D(P_t f), \\
\widetilde{D}( Q_t f) &= \partial_t( P_t f); \quad \text{and} \quad \\
\partial_t( \tilde{Q}_t f) &= -\tilde{D}( P_t f), \\
D( \tilde{Q}_t f) &= \partial_t( P_t f).
\end{align*}
\]

Moreover, $\partial_{tt}^2 Q_t f(n) + \Delta_d Q_t f(n) = 0$ and $\partial_{tt}^2 \tilde{Q}_t f(n) + \Delta_d \tilde{Q}_t f(n) = 0$.

(ii) We have

\[
\lim_{t \to 0} Q_t f(n) = \mathcal{R} f(n) \quad \text{and} \quad \lim_{t \to 0} \tilde{Q}_t f(n) = \tilde{\mathcal{R}} f(n),
\]

for $n \in \mathbb{Z}$. 

\[
Q_t f = \mathcal{R} P_t f \quad \text{and} \quad \tilde{Q}_t f = \tilde{\mathcal{R}} P_t f.
\]
In PDE’s.
- Analysis of the wave equation $\partial_t^2 + L = 0$.
- Work in progress with M. Kemppainen and P. Sjögren.
- Analysis of lateral operators. Work in progress with A. Bernardis, F. J. Martín-Reyes, P. Stinga, J. L. Torrea.
Other applications

In PDE’s.
- Analysis of the wave equation

$$\partial_t t + L = 0.$$ 

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In PDE’s.

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¡Muchas gracias por su atención!