Convex inequalities, isoperimetry and spectral gap

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   - Brunn-Minkowski inequality
   - C.Borell's inequality

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Part I. Convex inequalities

Convex inequalities: Hölder's

- Given $f, g : \mathbb{R}^n \to \mathbb{R}_+$ and $0 \leq \lambda \leq 1$

$$\int_{\mathbb{R}^n} f(x)^{1-\lambda} g(x)^{\lambda} dx \leq \left(\int_{\mathbb{R}^n} f(x) dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x) dx \right)^{\lambda}.$$

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- Case: $f = \chi_A$, $g = \chi_B$, $\qquad (A, B \subseteq \mathbb{R}^n)$,
  we have

$$|A \cap B|_n \leq |A|_n^{1-\lambda} |B|_n^{\lambda} \qquad \forall\, 0 \leq \lambda \leq 1$$

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$$|A \cap B|_n \le |A|_n^{1-\lambda} |B|_n^{\lambda} \qquad \forall\, 0 \le \lambda \le 1$$

$$\Updownarrow$$

$$|A \cap B|_n \le \min\{|A|_n, |B|_n\}.$$

Reverse

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$$\min\{|A|_n, |B|_n\} \not\leq |A \cap B|_n$$

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However ... Instead of

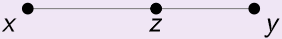
$$\int f^{1-\lambda} g^{\lambda}$$

we can use the sup-convolution

$$\int f^{1-\lambda} *_{\text{sup}} g^{\lambda}$$

Given $f, g : \mathbb{R}^n \to \mathbb{R}_+$ and $0 \leq \lambda \leq 1$

$$f^{1-\lambda} *_{\sup} g^{\lambda}(z) := \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g^{\lambda}(y)$$



Given $f, g : \mathbb{R}^n \to \mathbb{R}_+$ and $0 \leq \lambda \leq 1$

$$f^{1-\lambda} *_{\text{sup}} g^\lambda(z) := \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g^\lambda(y)$$

This function is not necessarily measurable, but we can consider its exterior integral:

$$\int_{\mathbb{R}^n}^* f^{1-\lambda} *_{\text{sup}} g^\lambda(z) dz = \inf \left\{ \int_{\mathbb{R}^n} h(z) dz : f^{1-\lambda} *_{\text{sup}} g^\lambda(z) \leq h(z) \right\}$$

Prekopa-Leindler's inequality

Let $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ such that, for some $0 \leq \lambda \leq 1$,

$$f(x)^{1-\lambda} g(y)^{\lambda} \leq h((1-\lambda)x + \lambda y) \qquad \forall\, x, y \in \mathbb{R}^n$$

$$\Downarrow$$

$$\left(\int_{\mathbb{R}^n} f(x)dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)dy \right)^{\lambda} \leq \int_{\mathbb{R}^n} h(z)dz$$

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Hence, reverse Hölder's inequality

$$\left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)dy\right)^\lambda \leq \int_{\mathbb{R}^n}^* f^{1-\lambda} *_{\sup} g^\lambda(z)dz$$

Proof: Dimension $n = 1$

Let $A, B \subseteq \mathbb{R}$ be non-empty compact sets. Then

$$|A + B|_1 \geq |A|_1 + |B|_1$$

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Let $A, B \subseteq \mathbb{R}$ be non-empty compact sets. Then

$$|A + B|_1 \geq |A|_1 + |B|_1$$

Indeed,

$$A + B \supseteq (\min A + B) \cup (A + \max B)$$

and

$$(\min A + B) \cap (A + \max B) = \min A + \max B,$$

$$\Downarrow$$

$$|A + B|_1 \geq |\min A + B|_1 + |A + \max B|_1 = |A|_1 + |B|_1$$

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$$|A + B|_1 \geq |A|_1 + |B|_1$$

W.l.o.g. $\|f\|_\infty = \|g\|_\infty = 1$.
Given $0 \leq t < 1$, since

$$\{x \in \mathbb{R} : h(x) \geq t\} \supseteq (1 - \lambda)\{x \in \mathbb{R} : f(x) \geq t\} + \lambda\{x \in \mathbb{R} : g(x) \geq t\}$$

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$$\begin{aligned}\int_{\mathbb{R}} h(x)dx &\geq \int_0^1 |\{h \geq t\}|dt \geq (1-\lambda)\int_0^1 |\{f \geq t\}|dt + \lambda \int_0^1 |\{g \geq t\}|dt \\ &\geq \text{ (by the arithmetic-geometric mean inequality)} \\ &\geq \left(\int_{\mathbb{R}} f(x)dx\right)^{1-\lambda}\left(\int_{\mathbb{R}} g(x)dx\right)^{\lambda}\end{aligned}$$

Induction for $n > 1$

Fix $x_1 \in \mathbb{R}$, let $f_{x_1} : \mathbb{R}^{n-1} \to [0, \infty)$ be $f_{x_1}(x_2, \ldots, x_n) = f(x_1, \ldots, x_n)$.
Whenever $z_1 = (1 - \lambda)x_1 + \lambda y_1$,

$$h_{z_1}((1-\lambda)(x_2, \ldots, x_n) + \lambda(y_2, \ldots, y_n)) \geq f_{x_1}(x_2, \ldots, x_n)^{1-\lambda} g_{y_1}(y_2, \ldots, y_n)^{\lambda}$$

for all $(x_2, \ldots, x_n), (y_2, \ldots, y_n) \in \mathbb{R}^{n-1}$.

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for all $(x_2, \ldots, x_n), (y_2, \ldots, y_n) \in \mathbb{R}^{n-1}$.
By the induction hypothesis

$$\int_{\mathbb{R}^{n-1}} h_{z_1}(\overline{z}) d\,\overline{z} \geq \left(\int_{\mathbb{R}^{n-1}} f_{x_1}(\overline{x}) d\,\overline{x} \right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_{y_1}(\overline{y}) d\,\overline{y} \right)^{\lambda}$$

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Applying again the inequality for $n = 1$ and Fubini's theorem we obtain the result.

Consequences. 1. Case: $f = \chi_A$, $g = \chi_B$, $A, B \subseteq \mathbb{R}^n$

Brunn-Minkowski inequality

Let A, B two Borel sets in $\mathbb{R}^n$. For any $0 \leq \lambda \leq 1$

$$|A|^{1-\lambda}|B|^{\lambda} \leq |(1-\lambda)A + \lambda B| \tag{1}$$

or equivalently

$$|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} \leq |A + B|^{\frac{1}{n}} \tag{2}$$

whenever $A \neq \emptyset \neq B$.

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- (1) $\Longrightarrow$ (2)

$$A' = A/|A|^{\frac{1}{n}} \qquad B' = B/|B|^{\frac{1}{n}} \qquad \lambda = |B|^{\frac{1}{n}}/(|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}})$$

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- (2) $\Longrightarrow$ (1)

$$|(1-\lambda)A + \lambda B|^{\frac{1}{n}} \geq (1-\lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}} \geq |A|^{\frac{1-\lambda}{n}}|B|^{\frac{\lambda}{n}}.$$

(by the arithmetic-geometric mean inequality)

2. Log-concave measures

A measure (or probability measure) $d\mu(x)$ in $\mathbb{R}^n$ is log-concave if

$$d\mu(x) = e^{-V(x)}dx$$

where $V : \mathbb{R}^n \to (-\infty, \infty]$ is a convex funtion.

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Examples

- Lebesgue measure
- uniform measure on K, convex body in $\mathbb{R}^n$ (compact, convex with non empty interior)
- exponential measure, $d\mu(x) = e^{-|x|}dx$
- Gaussian measure, $d\mu(x) = \frac{1}{(\sqrt{2\pi})^n} \exp\left(-\frac{|x|^2}{2}\right) dx$

Brunn-Minkowski inequality for log-concave probabilities

Any log-concave probability μ on $\mathbb{R}^n$ satisfies Brunn-Minkowski inequality *i.e.*,

$$\mu((1-\lambda)A+\lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}$$

for any $A, B \subseteq \mathbb{R}^n$ borelians and any $0 \leq \lambda \leq 1$.

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Proof.

We take $f(x) = \chi_A(x)e^{-V(x)}$, $g(y) = \chi_B(y)e^{-V(y)}$ and

$$h(z) = \chi_{(1-\lambda)A+\lambda B}(z)e^{-V(z)}$$

then we apply Prekopa-Leindler inequality. □

3. Isoperimetric inequality in $\mathbb{R}^n$

$$\frac{|\partial A|^{\frac{1}{n-1}}}{|A|^{\frac{1}{n}}} \geq \frac{|S^{n-1}|^{\frac{1}{n-1}}}{|B|^{\frac{1}{n}}}$$

where $A \subseteq \mathbb{R}^n$ is a bounded borelian

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$$|\partial A| = \liminf_{t \to 0} \frac{|A^t| - |A|}{t}$$

$$A^t = \{x \in \mathbb{R}^n; d(x, A) \leq t\} = A + tB$$

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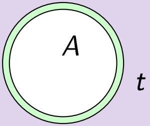

Proof.

$$\begin{aligned} |A^t| - |A| &= |A + tB| - |A| \\ &\geq \left(|A|^{\frac{1}{n}} + t\omega_n^{\frac{1}{n}} \right)^n - |A| \\ &= nt|A|^{\frac{n-1}{n}} \omega_n^{\frac{1}{n}} + o(t) \end{aligned}$$

Hence

$$\begin{aligned} |\partial A| &= \liminf_{t \to 0} \frac{|A^t| - |A|}{t} \\ &\geq n|A|^{\frac{n-1}{n}} \omega_n^{\frac{1}{n}} \end{aligned}$$

and then

$$\frac{|\partial A|^{\frac{1}{n-1}}}{|A|^{\frac{1}{n}}} \geq \frac{|S^{n-1}|^{\frac{1}{n-1}}}{|B|^{\frac{1}{n}}}$$

$\square$

C. Borell's inequality

Let μ be a log-concave probability in $\mathbb{R}^n$. Then for all symmetric convex sets $A \subseteq \mathbb{R}^n$ with $\mu(A) \geq \theta \geq \frac{1}{2}$

$$\mu(tA)^c \leq \theta \left(\frac{1-\theta}{\theta} \right)^{1+\frac{t}{2}} \qquad \forall\, t > 1$$

For instance, if $\mu(A) \geq 2/3$,

$$\mu(tA)^c \leq \frac{1}{2} \exp\left(-\frac{t \log 2}{2} \right) \qquad \forall\, t > 1$$

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- There is exponential decay of the mass for $(t > 1)$-dilations of A symmetric, with absolute constants

Proof.

It is a consequence of

$$A^c \supseteq \frac{2}{t+1}(tA)^c + \frac{t-1}{t+1}A$$

and Brunn-Minkowski

$$1-\theta \geq \mu(A^c) \geq \mu((tA)^c)^{\frac{2}{t+1}}\mu(A)^{\frac{t-1}{t+1}}$$

□

Reverse Hölder and exponential decay of seminorms

There exist a absolute constants $C_1, C_2 > 0$ such that for any log-concave probability on $\mathbb{R}^n$ and for any seminorm $f : \mathbb{R}^n \to [0, \infty)$ we have

i) $$\left(\int f^p d\mu\right)^{\frac{1}{p}} \leq C_1 p \int f d\mu, \qquad \forall\, p > 1$$

ii) $$\mu\left\{x \in \mathbb{R}^n : f(x) \geq C_2 t \int f d\mu\right\} \leq 2\exp\left(-t \log 2\right), \qquad \forall\, t > 0.$$

Proof of i)

Since any seminorm is integrable we can assume $\int f d\mu = 1$. Let $A = \{f < 3\}$. By Markov's inequality $\mu(A) \geq 2/3$.

$$\mu\{f \geq 3t\} = \mu(tA)^c \leq \frac{1}{2}\exp\left(-t\frac{\log 2}{2}\right), \qquad t > 1$$

Let $p > 1$

$$\int f^p d\mu = \int_0^3 p\, t^{p-1}\mu\{f > t\}dt + \int_3^\infty p\, t^{p-1}\mu\{f > t\}dt$$
$$\leq 3^p + 3^p \int_1^\infty p\, s^{p-1}e^{-2s}ds \leq (C_1 p)^p$$

for some absolute constant $C_1 > 0$ and i) follows.

Proof of ii)

Assume $\int f d\mu = 1$.
By Markov's inequality, taking $p = t \geq 1$

$$\mu\{f > 2C_1 t\} = \mu\left\{\frac{f}{2C_1 t} > 1\right\} \leq \int \frac{f^t}{(2C_1 t)^t} d\mu$$
$$\leq \frac{(C_1 t)^t}{(2C_1 t)^t} = e^{-t \log 2}$$

For $0 < t \leq 1$, the trivial bound 1 makes the job.

Moments

Case $f(x) = |x|$, we use $\mathbb{E}_\mu |x|^p = \int |x|^p d\mu$

- $$(\mathbb{E}_\mu |x|^p)^{\frac{1}{p}} \leq C_1 p \, \mathbb{E}_\mu |x| \qquad \forall \, p > 1$$

Moments

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- $$(\mathbb{E}_\mu |x|^p)^{\frac{1}{p}} \leq C_1 \, p \, \mathbb{E}_\mu |x| \qquad \forall \, p > 1$$

- $$\mu \left\{ |x| \geq C_2 \, t \mathbb{E}_\mu |x| \right\} \leq 2 \exp\left(-t\right), \qquad \forall \, t > 0.$$

- Paouris (2006) improved the inequality:

  $$(\mathbb{E}_\mu |x|^p)^{\frac{1}{p}} \leq C \max \left\{ \mathbb{E}_\mu |x|, p\lambda_\mu \right\}$$

  where $\lambda_\mu = \sup_{\theta \in S^{n-1}} \left(\mathbb{E}_\mu |\langle x, \theta \rangle|^2 \right)^{\frac{1}{2}}$

  $$\mu\{|x| \geq Ct\mathbb{E}_\mu |x|\} \leq \exp\left(-3 \, \frac{t\mathbb{E}_\mu |x|}{\lambda_\mu} \right)$$

  for $t \geq 1$ (it is stronger)