Maximal functions, weights and ae convergence of Poisson integrals

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(joint work with Harzstein, Signes, Torrea, Viviani)

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... and similarly for $p^*f(x) := \sup_{t>0} |e^{-t\sqrt{L}}f|(x)$...

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..."$\sup_{0<t<\infty}$" additionally implies $\lim_{t\to\infty} e^{t\Delta} f = 0$, *a.e.* x...

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  - non-constr proof (RdF): use VV ineq + abstract factorization thms...

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- Can do something similar for Hermite operator?

The Hermite operator: $L = -\Delta + |x|^2$

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- Bdedness of $h_a^* : L^p(w) \to L^p(v)$ holds similarly (with a few extra arguments...)

PROOFS: HERMITE-POISSON EQN

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...and $L^p(w) \subset L^1(\varphi) \Leftrightarrow \left\| w^{-\frac{1}{p}}\varphi \right\|_{p'} < \infty$

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THANKS