Two weight norm inequalities for fractional integrals and commutators

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Joint work with Kabe Moen, University of Alabama

Roll Tide!

Muchas gracias a los organizadores por la invitación.

Outline of Lectures

- Dyadic operators
- Digression: One weight inequalities
- Testing conditions
- A_p bump conditions

Lecture 1:
Dyadic operators and one weight inequalities

Fractional integral operators

For $0 < \alpha < n$,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

$$[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x), \qquad b \in BMO$$

Basic facts

- I_α is positive
- For $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$,

$$I_\alpha : L^p \to L^q, \quad [b, I_\alpha] : L^p \to L^q$$

- If $p = 1$, $\quad I_\alpha : L^p \to L^{q,\infty}$
- If $f \in C_c^\infty$, then $\quad |f(x)| \le C(n) I_1(|\nabla f|)(x)$

Applications

- Sobolev embedding: $1 \le p < n$, $\|f\|_q \le C(n)\|\nabla f\|_p$

  If $p = 1$, use Maz'ya / Long-Nie technique

- (Fefferman-Phong) Schrödinger operator $L = -\Delta - v$ positive if

$$\int_{\mathbb{R}^n} |u|^2 v\, dx \le C|\nabla u|^2\, dx, \quad u \in C_c^\infty$$

- Regularity of weak solutions of elliptic PDEs

  See Chiarenza and Franciosi (1992); DCU, Moen, Rodney (2014)

The new philosophy

Anything you can do,

I can do better (dyadically)!

With apologies to Irving Berlin
("Annie Get Your Gun", 1946)

Dyadic grids

A collection of cubes $\mathcal{D}$ is a dyadic grid if:

- $Q \in \mathcal{D} \implies \ell(Q) = 2^k,\ k \in \mathbb{Z}$
- if $Q,\ P \in \mathcal{D} \implies Q \cap P \in \{Q, P, \emptyset\}$.
- $k \in \mathbb{Z} \implies \mathcal{D}_k = \{Q \in \mathcal{D}, \ell(Q) = 2^k\}$ is partition of $\mathbb{R}^n$.

Sparse sets

A set $\mathcal{S} \subset \mathcal{D}$ is sparse if for every $Q \subset \mathcal{S}$,

$$\left| \bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subset Q}} Q' \right| \leq \frac{1}{2}|Q|$$

Define

$$E(Q) = Q \setminus \bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subset Q}} Q'$$

Then sets $E(Q)$ are pairwise disjoint and

$$|Q| \leq 2|E(Q)|.$$

Sparse sets exist

Theorem (Calderón-Zygmund)

For $k \in \mathbb{Z}$ and $a \geq 2^{n+1}$,

$$\{x \in \mathbb{R}^n : M^d f(x) > a^k\} = \bigcup_j Q_j^k$$

and $\{Q_j^k\}$ is sparse.

Cubes and dyadic cubes

Lemma

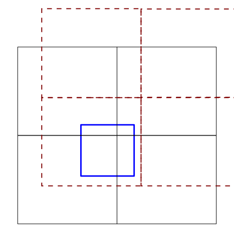
There exist $N = N(n)$ dyadic grids $\mathcal{D}^k$, $1 \le k \le N$, such that given any cube Q, there exists k and $P \in \mathcal{D}^k$ such that $Q \subset P$ and $\ell(P) \le 6\ell(Q)$.

$N = 3^n$: Christ, Garnett and Jones (attrib.);
$N = 2^n$, Hytönen and Pérez (2013);
$N = n + 1$, Conde (2012)

proof (sketch for $N = 3^n$)

Define dyadic grids

$$\mathcal{D}^t = \{2^j([0,1)^n + m + t) : j \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad t \in \{0, \pm 1/3\}^n$$



Fix cube Q, and j such that $\frac{2^j}{3} \le \ell(Q) < \frac{2^{j+1}}{3}$. At most 2^n dyadic cubes of sidelength 2^j intersect Q. Let P have largest intersection. Translate P distance $\frac{2^j}{3}$ in directions parallel to coordinate axes towards closest face of Q.

Two dyadic operators

Notation: $\langle f \rangle_Q = ⨍_Q f(y)\, dy$, $\langle f \rangle_{Q,\sigma} = \frac{1}{\sigma(Q)} \int_Q f(y)\sigma(y)\, dy$

Dyadic fractional integral

$$I_\alpha^{\mathcal{D}} f(x) = \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}} \langle f \rangle_Q \, \chi_Q(x)$$

Dyadic fractional maximal operator

$$M_\alpha^{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}} \langle f \rangle_Q \, \chi_Q(x)$$

Passing to dyadic operators

Theorem

There exists $N = N(n)$ dyadic grids $\mathcal{D}^k$ such that for any non-negative function f,

$$c(n,\alpha) I_\alpha^{\mathcal{D}^k} f(x) \le I_\alpha f(x) \le C(n,\alpha) \sup_k I_\alpha^{\mathcal{D}^k} f(x)$$

$$M_\alpha^{\mathcal{D}^k} f(x) \le M_\alpha f(x) \le C(n,\alpha) \sup_k M_\alpha^{\mathcal{D}^k} f(x)$$

Proof (sketch for I_α)

$$I_\alpha f(x) \lesssim \sum_{j\in\mathbb{Z}} 2^{j(\alpha-n)} \int_{Q(x,2^j)\setminus Q(x,2^{j-1})} f(y)\,dy$$

$$\lesssim \sum_{j\in\mathbb{Z}} \sum_{k=1}^{N} \sum_{\substack{Q\in\mathcal{D}^k\\ \ell(Q)\approx 2^j}} |Q|^{\frac{\alpha}{n}} \fint_Q f(y)\,dy\,\chi_Q(x)$$

$$\lesssim \sum_{k=1}^{N} I_\alpha^{\mathcal{D}^k} f(x)$$

Dyadic commutators

Theorem

There exists $N = N(n)$ dyadic grids $\mathcal{D}^k$ such that for any non-negative function f and $b \in BMO$,

$$|[b, I_\alpha]f(x)| \le \sum_{k=1}^{N} \sum_{Q\in\mathcal{D}^k} |Q|^{\frac{\alpha}{n}} \fint_Q |b(y)-b(x)| f(y)\,dy\,\chi_Q(x).$$

Implicit in DCU-Moen (2012)

Sparse operators

Given $\mathcal{D}$ and a sparse subset $\mathcal{S}$, define

Sparse dyadic fractional integral

$$I_\alpha^{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} \langle f\rangle_Q\,\chi_Q(x)$$

Sparse linearization of dyadic maximal operator

$$L_\alpha^{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} \langle f\rangle_Q\,\chi_{E(Q)}(x)$$

Passing to sparse operators

Theorem

Given $\mathcal{D}$ and a non-negative function $f \in L_c^\infty$, there exist sparse sets $\mathcal{S} \subset \mathcal{D}$ such that

$$I_\alpha^{\mathcal{D}} f(x) \le C(n,\alpha) I_\alpha^{\mathcal{S}} f(x)$$

$$M_\alpha^{\mathcal{D}} f(x) \le C(n,\alpha) L_\alpha^{\mathcal{S}} f(x).$$

For $I_\alpha^{\mathcal{D}}$ implicit in Pérez (1994);
For $M_\alpha^{\mathcal{D}}$ implicit in Sawyer (1982)

Proof (sketch for I_α)

$$\mathcal{Q}_k = \{Q \in \mathcal{D} : \langle f\rangle_Q \approx a^k\}, \qquad a \geq 2^{n+1}$$
$$\mathcal{S}_k = \{P \in \mathcal{D} \text{ maximal} : \langle f\rangle_P > a^k\}$$

$\mathcal{S} = \bigcup \mathcal{S}_k$ is sparse.

$$\begin{aligned} I_\alpha^{\mathcal{D}} f(x) &= \sum_{Q\in\mathcal{D}} |Q|^{\frac{\alpha}{n}} \langle f\rangle_Q\, \chi_Q(x) \\ &\leq \sum_k a^{k+1} \sum_{P\in\mathcal{S}_k} \sum_{\substack{Q\in\mathcal{Q}_k \\ Q\subset P}} |Q|^{\frac{\alpha}{n}}\, \chi_Q(x) \\ &\leq C(n,\alpha) \sum_k a^{k+1} \sum_{P\in\mathcal{S}_k} |P|^{\frac{\alpha}{n}}\, \chi_P(x) \\ &\leq C(n,\alpha) I_\alpha^{\mathcal{S}} f(x) \end{aligned}$$

Introduction Dyadic operators Sparse operators One weight inequalities

The key idea

Hereafter, to prove any inequality for I_α or M_α it suffices to prove it for the corresponding sparse operator.

Morally, this is also true for $[b, I_\alpha]$.

Introduction Dyadic operators Sparse operators One weight inequalities

$A_{p,q}$ weights

For $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, we say $w \in A_{p,q}$ if

$$[w]_{A_{p,q}} = \sup_Q \left(\fint_Q w(x)^q \, dx \right)^{1/q} \left(\fint_Q w(x)^{-p'} \, dx \right)^{1/p'} < \infty.$$

Lemma

If $w \in A_{p,q}$, then w^q, $w^{-p'} \in A_\infty$.

Introduction Dyadic operators Sparse operators One weight inequalities

One weight norm inequalities

Theorem (Muckenhoupt, Wheeden (1974))

For $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, if $w \in A_{p,q}$, then

$$\left(\int_{\mathbb{R}^n} |M_\alpha f(x) w(x)|^q \, dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x) w(x)|^p \, dx \right)^{1/p}$$

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x) w(x)|^q \, dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x) w(x)|^p \, dx \right)^{1/p}$$

Proof for M_α

Fix $\mathcal{D}$ and $\mathcal{S} \subset \mathcal{D}$ sparse. Let $\sigma = w^{-p'} \in A_\infty$.

$$\begin{aligned}
\|(L_\alpha^{\mathcal{S}} f) w\|_q^q &= \sum_{Q\in\mathcal{S}} \left(|Q|^{\frac{\alpha}{n}} \langle f\rangle_Q\right)^q w^q(E_Q) \\
&\le \sum_{Q\in\mathcal{S}} \left(\sigma(Q)^{\frac{\alpha}{n}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\right)^q |Q|^{q\frac{\alpha}{n}-q} \sigma(Q)^{q-q\frac{\alpha}{n}} w^q(Q) \\
&\lesssim \sum_{Q\in\mathcal{S}} \left(\sigma(Q)^{\frac{\alpha}{n}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\right)^q \underbrace{|Q|^{-\frac{q}{p'}-1}\sigma(Q)^{\frac{q}{p'}} w^q(Q)}_{[w]_{A_{p,q}}^q} \sigma(E_Q) \\
&\lesssim \int_{\mathbb{R}^n} M_{\alpha,\sigma}^{\mathcal{D}}(f\sigma^{-1})^q \, d\sigma \\
&\lesssim \|fw\|_p^q.
\end{aligned}$$

Proof for I_α

Lemma

If $u \in A_\infty$, then for $0 < q < \infty$,

$$\int_{\mathbb{R}^n} |I_\alpha^{\mathcal{S}} f(x)|^q u \, dx \le C \int_{\mathbb{R}^n} |L_\alpha^{\mathcal{S}} f(x)|^q u \, dx$$

Muckenhoupt-Wheeden (1974): I_α and M_α via good-λ inequality.

2nd proof: Sharp function estimate

Lemma

If $u \in A_\infty$, then for $0 < q < \infty$,

$$\int_{\mathbb{R}^n} |f(x)|^q u \, dx \le C \int_{\mathbb{R}^n} |M^{\mathcal{D},\#} f(x)|^q u \, dx$$

Journé (1983) via good-λ inequality;
Lerner (2004), DCU-Martell-Pérez (2007) via atomic decomposition and extrapolation.

Pointwise estimate

Lemma

For $f \in L_c^\infty$,

$$M^{\mathcal{D},\#}(I_\alpha^{\mathcal{D}} f)(x) \le C(n,\alpha) M_\alpha^{\mathcal{D}} f(x).$$

Adams (1975) for $M^\#$, I_α and M_α

Proof of Lemma (sketch)

Fix $P \in \mathcal{D}$ and $x \in P$.

$$I_\alpha^{\mathcal{D}} f(x) = \sum_{Q \subseteq P} |Q|^{\frac{\alpha}{n}} \langle f \rangle_Q \chi_Q(x) + \sum_{P \subsetneq Q} |Q|^{\frac{\alpha}{n}} \langle f \rangle_Q \, \chi_Q(x)$$
$$\leq I_\alpha^{\mathcal{D}}(f\chi_P)(x) + c_P.$$

Therefore, by Kolmogorov's inequality,

$$⨍_P |I_\alpha^{\mathcal{D}} f(x) - c_P| \, dx \leq ⨍_P I_\alpha^{\mathcal{D}}(f\chi_P)(x) \, dx$$
$$\leq C(n,\alpha)|P|^{\frac{\alpha}{n}} ⨍_P f \, dy \leq C(n,\alpha) M_\alpha^{\mathcal{D}} f(x).$$

Commutators

Theorem

For $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, if $w \in A_{p,q}$ and $b \in BMO$, then

$$\left(\int_{\mathbb{R}^n} |[I_\alpha, b] f \, w|^q \, dx \right)^{1/q} \leq C \|b\|_{BMO} \left(\int_{\mathbb{R}^n} |f \, w|^p \, dx \right)^{1/p}$$

DCU-Moen (2012), using Cauchy integral formula argument of Chung-Pereyra-Pérez (2012).

End of Lecture 1
Questions?