Composition operators on Hardy spaces

Episode I

VI Curso Internacional de Análisis Matemático en Andalucía

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Program

1. Lecture 1
   - Classical Hardy spaces on $\mathbb{D}$
   - Composition operators
     - Boundedness
     - Compactness

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   - H^∞
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   - Carleson versus Nevanlinna

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   - Hardy-Orlicz spaces and their composition operators
   - Carleson versus Nevanlinna
3. Lecture 3
   - Schatten classes, approximation numbers
   - Absolutely summing composition operators
   - Some open problems...

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They are all Banach spaces...

Some (useful) properties...

- Every $f \in H^p$ has almost everywhere radial limit f^*

$$f^*(e^{it}) = \lim_{r\to 1^-} f\left(re^{it}\right).$$

It is known that $f^* \in L^p(\mathbb{T})$ and $\|f\|_{H^p} = \|f^*\|_{L^p(\mathbb{T})}$.

In fact, $f^* \in \{g \in L^p(\mathbb{T}) \mid \hat{g}(m) = 0 \text{ for every } m < 0\}$.

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- Conversely, if $g \in L^p(\mathbb{T})$, with $\hat{g}(m) = 0$ for every $m < 0$, the Poisson integral of g at point $z = re^{2i\pi\theta}$

$$P[g](z) = P_r * g(\theta) = \int_0^1 P_r(\theta - t) g\left(e^{2i\pi t}\right) dt,$$

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- We have $H^p \sim \{g \in L^p(\mathbb{T}) \mid \hat{g}(m) = 0 \text{ for every } m < 0\}$.

- Hence we will consider that a function $f \in H^p$ is defined, not only on $\mathbb{D}$, but on the whole $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$.

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  - On the Hilbert space H^2, the functional δ_z is associated to the reproducing kernel $w \in \overline{\mathbb{D}} \longmapsto \dfrac{1}{1 - \bar{z}w}$.
  - $\|\delta_z\|_{(H^p)^*} = \left(\dfrac{1}{1 - |z|^2}\right)^{1/p} \approx \dfrac{1}{(1 - |z|)^{1/p}}$.

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For the reverse inequality: consider $w \in \overline{\mathbb{D}} \mapsto \left(\frac{1}{1-\bar{z}w} \right)^{2/p}.$

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2. Using the Nevanlinna counting function N_φ.
3. The Carleson embedding point of view, in terms of Carleson measures.

Boundedness

Let us treat the case of the special (but important) case of Moebius transformations, which are automorphisms of the disk. Consider the Moebius transformation $q_a(z) = \dfrac{a - z}{1 - \bar{a}z}$, where $a \in \mathbb{D}$.

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For every polynomial f, we have

$$\|f \circ q_a\|_{H^p}^p = \int_{\mathbb{T}} |f(z)|^p \frac{1 - |a|^2}{|1 - \bar{a}z|^2}\, d\lambda \leq \frac{1 + |a|}{1 - |a|} \|f\|_{H^p}^p.$$

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Actually

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Hence if we prove that C_ϕ is bounded (with $\|C_\phi\| = 1$):

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Boundedness via the subordination principle

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ *be an analytic function with* $\varphi(0) = 0$, *and* $g\colon \mathbb{D} \to [0, +\infty)$ *a subharmonic function. We have for every* $r \in (0,1)$

$$\int_0^{2\pi} g\big(\varphi(re^{it})\big)\, dt \le \int_0^{2\pi} g\big(re^{it}\big)\, dt\,.$$

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Think to the Schwarz lemma !

$$\frac{1}{2\pi}\int_0^{2\pi} g\big(\varphi(re^{it})\big)\, dt \quad \le \frac{1}{2\pi}\int_0^{2\pi} G\big(\varphi(re^{it})\big)\, dt =$$

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Do not forget that $G \circ \varphi$ is harmonic !

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The boundedness is proved !!

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$$\|f\|_2^2 = \big|f(0)\big|^2 + 2 \int_{\mathbb{D}} |f'|^2 \log \frac{1}{|z|} \, d\mathcal{A}$$

implies again the boundedness of C_φ is bounded on H^2.

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Indeed:

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Point out that

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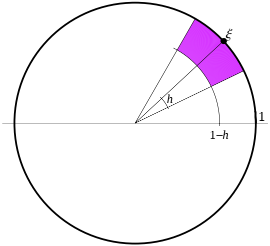

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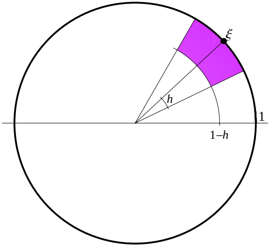

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The compactness of C_φ on H^p is proved.

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It can be also written

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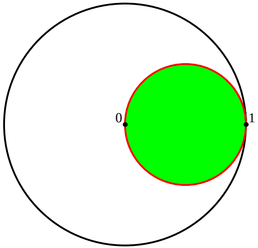
Compactness on Hardy spaces: two examples

$$\varphi(z) = \frac{1+z}{2}$$



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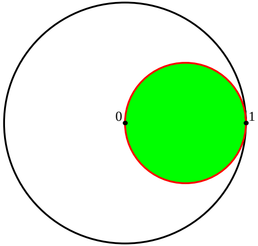

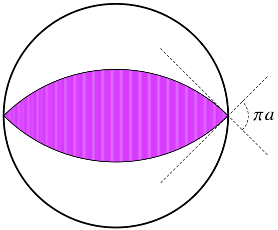

Lens map ($0 < a < 1$)

Some characterizations of compactness

Theorem(Power 80, Mac-Cluer 85)

C_φ is compact *if and only if* λ_φ is a vanishing Carleson measure i.e.

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} \lambda_\varphi\big(W(\xi, h)\big) = o\,(h) \qquad \text{when } h \to 0$$

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Actually:

$$\|C_\varphi\|_e = \limsup_{|w| \to 1^-} \left(\frac{N_\varphi(w)}{1 - |w|} \right)^{1/2} = \lim_{h \to 0} \left(\frac{\nu_\varphi(h)}{h} \right)^{1/2}.$$

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(Ackeroyd '10)

$$\|C_\varphi\|_e = \limsup_{|a|\to 1^-} \left\| C_\varphi\left(\frac{k_a}{\|k_a\|_{H^2}}\right)\right\|_{H^2}$$

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for any $r \in (0,1)$.

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$$C_\varphi \text{ is compact on } H^p \Longleftrightarrow \lim_{|z|\to 1^-} \frac{1-|\varphi(z)|}{1-|z|} = \infty$$

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The converse is false in general: McCluer-Shapiro ('86) constructed inner functions φ admitting no angular derivatives at any point of the circle.

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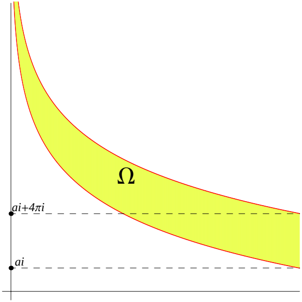
This allowed MacCluer and Shapiro ('86) to construct an example of a (finitely valent) symbol $\varphi\colon \mathbb{D} \to \mathbb{D}$ such that C_φ is compact, but φ is onto: $\varphi(\mathbb{D}) = \mathbb{D}$.

Construction of the McCluer-Shapiro's example

Let $g\colon (0,+\infty) \to \mathbb{R}$ be a continuous decreasing function such that $\lim_{x\to 0^+} g(x) = +\infty$ (for instance $g(x) = 1/x$). And consider the domain

$$\Omega = \{x + iy : g(x) < y < g(x) + 4\pi\}$$



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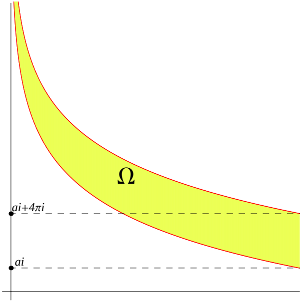

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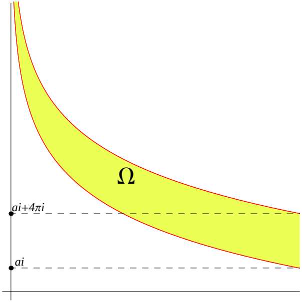

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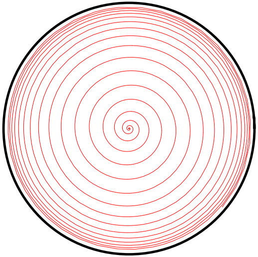

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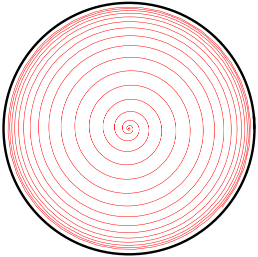

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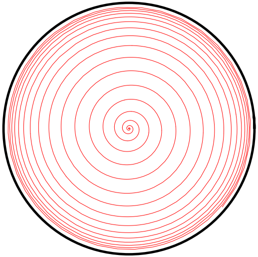

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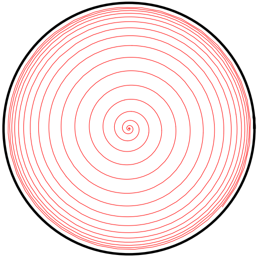

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φ is onto and $C_\varphi = C_{\varphi_1} \circ C_{Q_a}$ is compact.

Merci !