WEIGHTED INEQUALITIES FOR ONE-SIDED OPERATORS

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What does it mean?

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $T : X \to X$ be a measurable transformation.
The orbit of a point $x \in X$ is

$$\{x, Tx, T^2x, \dots, T^nx, \dots\}$$

where $T^n = T \circ T \circ \dots \circ T$ is the n-th iterated of T.
We could be interested in knowing how often the orbit of a point x enters in a measurable set A

Mean frequency

$$\frac{1}{n+1} \sum_{i=0}^{n} \chi_A(T^i x),$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \chi_A(T^i x).$$

What does it mean?

More generally, for any measurable function f, study

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(T^i x)\,.$$

To face this problem one considers the associated maximal operator

$$\mathcal{M}f(x) = \sup_{n\in\mathbb{N}} \frac{1}{n+1}\sum_{i=0}^{n} |f(T^i x)|\,.$$

What does it mean?

The analogous situation in the continuous case:
Semiflow of measurable transformations

$$\{T_t : t \geq 0\}, T_t : X \to X .$$

The associated maximal operator in this case is

$$\mathcal{M}^+ f(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(T_t x)| \, dt.$$

If we are in $\mathbb{R}$ and the semiflow is given by $T_t x = x + t$ then the maximal operator is

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(x+t)| \, dt = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| \, dt.$$

What does it mean?

In general a one-sided operator (in $\mathbb{R}$) is an operator T such that the value of $Tf(x)$ depends only on the values of f in $[x, \infty)$ or in $(-\infty, x]$.

In the first case, when "looking to the future", write T^+.

In the second case, when "looking to the past", write T^-.

Examples

The Hardy operator and its adjoint

$$Pf(x) = \int_{-\infty}^{x} f(y)dy\,, \qquad Qf(x) = \int_{x}^{\infty} f(y)dy$$

The Hardy averaging operator defined for functions in $(0, \infty)$

$$Hf(x) = \frac{1}{x}\int_{0}^{x} f(y)dy$$

The Riemann-Liouville and the Weyl integral operators, $0 < \alpha < 1$

$$I_\alpha^- f(x) = \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}}dy, \quad I_\alpha^+ f(x) = \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}}dy$$

Examples

For $f \in L^1_{loc}(\mathbb{R})$ and $n \in \mathbb{Z}$

$$A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y)dy$$

Discrete square function

$$Sf(x) = \left(\sum_{n\in\mathbb{Z}} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{1/2}$$

Differential transform

$$Df(x) = \sum_{n\in\mathbb{Z}} \nu_n (A_n f(x) - A_{n-1} f(x)),$$

where $\{\nu_n\}_{n\in\mathbb{Z}}$ is a bounded sequence.

Examples

One-sided Calderón-Zygmund singular integrals

They are singular integrals associated to a Calderón-Zygmund kernel K with support on $(-\infty, 0)$ or $(0, \infty)$

$$\textit{supp } K \subset (-\infty, 0) \mapsto T^+, \qquad \textit{supp } K \subset (0, \infty) \mapsto T^-$$

$$K(x) = \frac{1}{x} \frac{\sin(\log|x|)}{\log|x|} \chi_{(-\infty,0)}(x)$$

$$T^+ f(x) = \lim_{\varepsilon \to 0^+} \int_{x+\varepsilon}^{\infty} f(y) K(x-y) dy$$

Examples

One-sided Hardy-Littlewood maximal functions

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|dy, \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|dy$$

Weights for the one-sided Hardy-Littlewood Maximal Operator

Andersen, Sawyer, Martín-Reyes and de la Torre.

Let $1 \leq p < \infty$ and let u, v nonnegative measurable functions locally integrable in $\mathbb{R}$. The following conditions are equivalent:

Weak type inequality

There exists $C > 0$ such that for all $\lambda > 0$ and $f \in L^p(v)$

$$\int_{\{x \in \mathbb{R} : M^+ f(x) > \lambda\}} u(x) dx \leq \frac{C}{\lambda^p} \int_{\mathbb{R}} |f(x)|^p v(x) dx \, .$$

Weights for the one-sided Hardy-Littlewood Maximal Operator

A_p^+ condition, $p > 1$

There exists $C > 0$ such that for all $h > 0$ and $x \in \mathbb{R}$

$$\left(\frac{1}{h}\int_{x-h}^{x} u\right)^{1/p}\left(\frac{1}{h}\int_{x}^{x+h} v^{1-p'}\right)^{1/p'} \leq C.$$

A_1^+ condition

There exists $C > 0$ such that

$$M^- u(x) \leq Cv(x)\,, \quad \text{a.e.}$$

Weights for the one-sided Hardy-Littlewood Maximal Operator

Let $1 < p < \infty$. The following conditions are equivalent:

Strong type inequality

There exists $C > 0$ such that for any $f \in L^p(v)$,

$$\int_{\mathbb{R}} (M^+ f(x))^p u(x)\, dx \leq C \int_{\mathbb{R}} |f(x)|^p v(x)\, dx\,.$$

Sawyer's S_p^+ condition

There exists $C > 0$ such that

$$\int_I (M^+(\sigma\chi_I)(x))^p u(x)\, dx \leq C \int_I \sigma(x)\, dx < \infty\,,$$

for all intervals $I = (a, b)$ such that $\int_{-\infty}^a u(x)\, dx > 0$, where $\sigma = v^{1-p'}$ and $p + p' = pp'$.

Weights for the one-sided Hardy-Littlewood Maximal Operator

If $u = v$ the previous conditions are equivalent to

A_p^+ condition

There exists $C > 0$ such that for all $h > 0$ and $x \in \mathbb{R}$

$$\left(\frac{1}{h} \int_{x-h}^{x} u \right)^{1/p} \left(\frac{1}{h} \int_{x}^{x+h} u^{1-p'} \right)^{1/p'} \leq C\,.$$

There are similar conditions and results for M^-.

$$A_p = A_p^+ \cap A_p^-$$

$$A_p \subsetneq A_p^+ \qquad \text{y} \qquad A_p \subsetneq A_p^-\,.$$

Coifman type inequalities

Let $A_\infty^+ = \cup_{q\geq 1} A_q^+$. For $0 < p < \infty$ and $w \in A_\infty^+$

One-sided C-Z singular integrals, Aimar, Forzani, Martín-Reyes

$$\int_{\mathbb{R}} |T^+ f|^p w \leq C \int_{\mathbb{R}} (M^+ f)^p w$$

Discrete square function, L., Riveros, de la Torre

$$\int_{\mathbb{R}} |Sf|^p w \leq C \int_{\mathbb{R}} (M^+ \circ M^+ f)^p w$$

Differential transform, L., Martell, Riveros, de la Torre

$$\int_{\mathbb{R}} |Df|^p w \leq C \int_{\mathbb{R}} (M^+ \circ M^+ \circ M^+ f)^p w$$

One-sided Hardy-Littlewood maximal functions in $\mathbb{R}^n$

Natural generalization of M^+ in $\mathbb{R}^n$: for $x = (x_1, x_2, \ldots, x_n)$, define

One-sided maximal function in $\mathbb{R}^n$

$$M^{++\cdots+}f(x_1, x_2, \ldots, x_n) = \sup_{h>0} \frac{1}{h^n} \int_{Q_x(h)} |f(y)|\, dy\,,$$

where $Q_x(h) = [x_1, x_1 + h) \times [x_2, x_2 + h) \times \cdots \times [x_n, x_n + h)$.

Weighted inequalities for $M^{++\cdots+}$ in $\mathbb{R}^n$ have not been characterized.

Weights for the one-sided HL maximal operator in $\mathbb{R}^n$

Forzani, Martín-Reyes and Ombrosi gave a characterization of the weak type (p, p) inequality for M^{++} but only in dimension $n = 2$.

Let $1 \leq p < \infty$ and let u, v be two weights. The following conditions are equivalent:

Weak type inequality

There exists $C > 0$ such that for any $\lambda > 0$ and $f \in L^p(v)$

$$\int_{\{x \in \mathbb{R}^2 : M^{++}f(x) > \lambda\}} u(x)dx \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^2} |f(x)|^p v(x)dx\,.$$

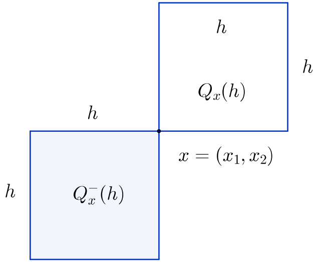
Weights for the one-sided HL maximal operator in $\mathbb{R}^n$

One-sided Muckenhoupt's type condition, $p > 1$

There exists $C > 0$ such that for all $x \in \mathbb{R}^2$ and all $h > 0$,

$$\frac{1}{h^2}\left(\int_{Q_x^-(h)} u\right)^{1/p}\left(\int_{Q_x(h)} v^{1-p'}\right)^{1/p'} < C,$$

where $Q_x^-(h) = [x_1 - h, x_1) \times [x_2 - h, x_2)$.



Weights for the one-sided HL maximal operator in $\mathbb{R}^n$

One-sided Muckenhoupt's type condition, $p = 1$

There exists $C > 0$ such that for any $h > 0$,

$$\frac{1}{h^2} \int_{Q_x^-(h)} u \leq Cv(x), \quad \text{c.t.p.} \quad x = (x_1, x_2).$$

The previous conditions are necessary for the weak type inequality in any dimension, but we don't know if they are sufficient.

Dyadic maximal operator in $\mathbb{R}^n$

Sawyer's proof that S_p condition characterizes de strong type inequality for the classical Hardy-Littlewood maximal operator requires to prove a similar result for the dyadic maximal operator

$$M_d f(x) = \sup_{x \in Q, Q \text{ dyadic}} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

and subsequently he uses an averaging argument due to Fefferman and Stein.

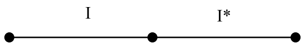
Then, in order to tackle the problem in the one-sided case, it is natural to study one-sided versions of this dyadic operator.

One-sided dyadic maximal operators

Some one-sided dyadic maximal operators have already been studied

Martín-Reyes and de la Torre, $n = 1$

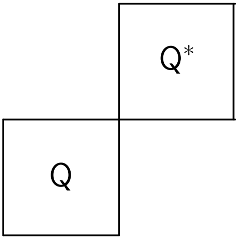
$$M_d^+ f(x) = \sup_{x \in I,\, I \text{ dyadic}} \frac{1}{|I^*|} \int_{I^*} |f(y)|dy.$$



One-sided dyadic maximal operators

Ombrosi, $n > 1$

$$M_d^{+\cdots+} f(x) = \sup_{x \in Q,\, Q \text{ dyadic}} \frac{1}{|Q^*|} \int_{Q^*} |f(y)| dy.$$



One-sided dyadic maximal operators

The previous operators do not satisfy the inequality

$$M_d^+ f \lesssim M_d f\,, \qquad M_d^{+\cdots+} f \lesssim M_d f\,,$$

where M_d is the classical dyadic maximal operator,

$$M_d f(x) = \sup_{x\in Q, Q \text{ dyadic}} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

We think that an optimal dyadic maximal operator should satisfy

$$M_d^{+\cdots+} f \lesssim M_d f \qquad \text{y} \qquad M_d^{+\cdots+} f \lesssim M^{+\cdots+} f$$

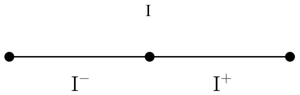
One-sided dyadic maximal operators

Joint work with F.J. Martín-Reyes.

We have got a one-sided dyadic maximal operator satisfying these conditions in $\mathbb{R}$:

One-sided dyadic maximal operator in $\mathbb{R}$

$$M_d^+ f(x) = \sup_{I \text{ dyadic},\, x \in I^-,} \frac{1}{|I^+|} \int_{I^+} |f(y)|dy.$$



One-sided dyadic maximal operators

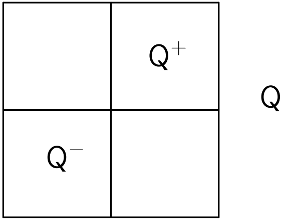
We have characterized the good weights for this dyadic operator and we have got an inequality in means analogous to the classical one.

As a consequence we get the characterization of the good weights for the one-sided maximal operator in $\mathbb{R}$, M^+.

We have tried to generalize it to $\mathbb{R}^n$, but we have not achieved our main goal.

One-sided dyadic maximal operators

$$M_d^{+\cdots+}f(x) = \sup_{Q \text{ dyadic},\, x \in Q^-} \frac{1}{|Q^+|} \int_{Q^+} |f(y)|dy.$$



One-sided dyadic maximal operators

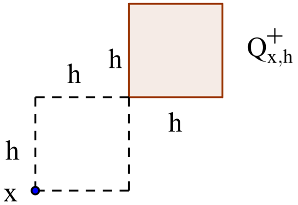
$$M_d^{+\cdots+}f(x) = \sup_{Q \text{ dyadic}, \, x \in Q^-} \frac{1}{|Q^+|} \int_{Q^+} |f(y)|dy.$$

- It generalizes the case $n = 1$.
- It satisfies $M_d^{+\cdots+}f \lesssim M_d f$ and $M_d^{+\cdots+}f \lesssim M^{+\cdots+}f$.
- Problem: We have not been able to prove an inequality in means. We don't get any result for $M^{+\cdots+}$.

Another one-sided maximal operator

$$N^{+\cdots+}f(x_1,\ldots,x_n) = \sup_{h>0} \frac{1}{|Q^+_{x,h}|} \int_{Q^+_{x,h}} |f(y)|dy\,,$$

where $Q^+_{x,h} = [x_1+h, x_1+2h) \times \cdots \times [x_n+h, x_n+2h)$.



Ombrosi proved that for $1 < p < \infty$, the condition

$(u, v) \in A_p^+$

There exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and all $h > 0$,

$$\frac{1}{h^n}\left(\int_{Q_x^-(h)} u\right)^{1/p}\left(\int_{Q_x(h)} v^{1-p'}\right)^{1/p'} < C$$

implies the weak type inequality for $N^{+\cdots+}$

Weak type inequality

There exists $C > 0$ such that for any $\lambda > 0$ and $f \in L^p(v)$

$$\int_{\{x\in\mathbb{R}^n : N^{+\cdots+}f(x)>\lambda\}} u(x)dx \leq \frac{C}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\,.$$

Lerner and Ombrosi proved that for $n = 2$ and $u = v$
A_p^+ condition implies that N^{++} is bounded from $L^p(u)$ into $L^p(u)$.

Berkovits extends this result for $n \geq 3$.

For $k \in \mathbb{Z}$ let

$$N_k^{+\cdots+} f(x) = \sup_{0<h<2^k} \frac{1}{|Q_{x,h}^+|} \int_{Q_{x,h}^+} |f(y)| dy \,.$$

Fefferman-Stein type inequality in means

There exists $C > 0$ such that

$$N_k^{+\cdots+} f(x) \leq \frac{C}{2^{kn}} \int_{(0,2^{k+4}]\times\cdots\times(0,2^{k+4}]} (\tau_{-t} \circ M_d^{+\cdots+} \circ \tau_t) f(x) dt,$$

where $\tau_t g(x) = g(x-t)$.

Weights for the one-sided dyadic maximal operator

Let $1 \leq p < \infty$. We say that the pair of weights (u, v) belongs to $A^+_{p,d}$ if

$A^+_{p,d}$, $\quad 1 < p < \infty$

there exists $C > 0$ such that for all dyadic cubes Q

$$\frac{1}{|Q|}\left(\int_{Q^-} u\right)^{1/p}\left(\int_{Q^+} v^{1-p'}\right)^{1/p'} \leq C$$

$A^+_{1,d}$

there exists $C > 0$ such that for all dyadic cubes Q

$$\frac{1}{|Q^-|}\int_{Q^-} u \leq Cv(x)\,,\ \text{c.t.p. } x \in Q^+\,.$$

Weights for the one-sided dyadic maximal operator

Theorem

Let $1 \leq p < \infty$. The following conditions are equivalent

- $(u, v) \in A^+_{p,d}$
- There exists $C > 0$ such that for all $\lambda > 0$ and $f \in L^p(v)$

$$\int_{\{x: M_d^{+\cdots+} f(x) > \lambda\}} u(x)dx \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x)dx \,.$$

Weights for the one-sided dyadic maximal operator

Let $1 < p < \infty$ and let u, v be two weights. The following conditions are equivalent

Strong type inequality

There exists $C > 0$ such that for all $f \in L^p(v)$,

$$\int_{\mathbb{R}^n} (M_d^{+\cdots+} f(x))^p u(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx.$$

Sawyer's type condition $S_{p,d}^{+}$

There exists $C > 0$ such that

$$\int_{Q^- \cup Q^+} (M_d^{+\cdots+}(\sigma \chi_{Q^+})(x))^p u(x)\, dx \leq C \int_{Q^+} \sigma(x)\, dx < \infty,$$

for all dyadic cubes Q with $\int_{Q^-} u > 0$, where $\sigma = v^{1-p'}$.

Weights for the one-sided dyadic maximal operator

If $u = v$ the previous inequalities are equivalent to

Muckenhoupt's type $A_{p,d}^{+}$ condition

There exists $C > 0$ such that for all dyadic cubes Q,

$$\frac{1}{|Q|}\left(\int_{Q^-} u\right)^{1/p}\left(\int_{Q^+} u^{1-p'}\right)^{1/p'} \leq C\,.$$

Boundedness of $N^{+\cdots+}$

From the inequality in means and the previous result we get that if $1 < p < \infty$ and u is a weight satisfying

A_p^+ condition

There exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and all $h > 0$,

$$\frac{1}{h^n}\left(\int_{Q_x^-(h)} u\right)^{1/p}\left(\int_{Q_x(h)} u^{1-p'}\right)^{1/p'} < C\,.$$

Then $N^{+\cdots+}$ is bounded from $L^p(u)$ into $L^p(u)$.

This extends Ombrosi's result to dimension $n \geq 3$ with a different proof to the one proposed by Berkovits.

Boundedness of $N^{+\cdots+}$

Let $1 < p < \infty$. If u, v are two weights satisfying

S_p^+ condition

There exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and all $h > 0$,

$$\int_{Q_{x,h}^- \cup Q_{x,h}} (M^{+\cdots+}(\sigma\chi_{Q_{x,h}}))^p u \leq C \int_{Q_{x,h}} \sigma < \infty,$$

whenever $\int_{Q_{x,h}^-} u > 0$, where $\sigma = v^{1-p'}$

then $N^{+\cdots+}$ is bounded from $L^p(u)$ into $L^p(v)$.

$n = 1$

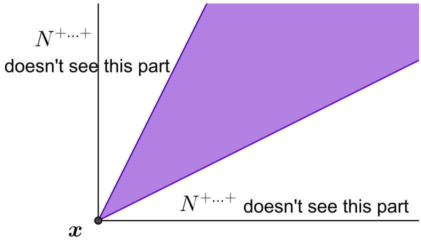
M^+ and N^+ are equivalent

$$M^+ f \lesssim N^+ f \qquad \text{and} \qquad N^+ f \lesssim M^+ f$$

$n > 1$

$M^{+\cdots+}$ and $N^{+\cdots+}$ are not equivalent

$$N^{+\cdots+} f \leq M^{+\cdots+} f \qquad \text{but} \qquad M^{+\cdots+} f \not\leq C N^{+\cdots+} f$$



Thanks for your attention!

¡Muchas gracias!