# The optimal modulus of convexity of a super-reflexive Banach space

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- As we will consider several equivalent norms on $X$ we prefer to write $\delta_{\|.\|}(t)$.

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That suggests us a way to compare the degree of uniform convexity between spaces, or norms (even inside the same space).

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## Theorem (James, Enflo, Pisier)

A Banach space has an equivalent uniformly convex norm if and only if it is superreflexive.

Moreover, we can take the norm such that $\delta(\varepsilon) \geq c\varepsilon^p$ for some $c > 0$ and $p \geq 2$.

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Pisier's result implies that if $X$ is super-reflexive, then there exists an equivalent norm $|||\cdot|||$ on $X$ and $p \geq 2$ such that

$$t^p \preceq \delta_{|||\cdot|||}(t)$$

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We do not know if $\mathfrak{N}_X(t)^{-1}$ is always a maximum.

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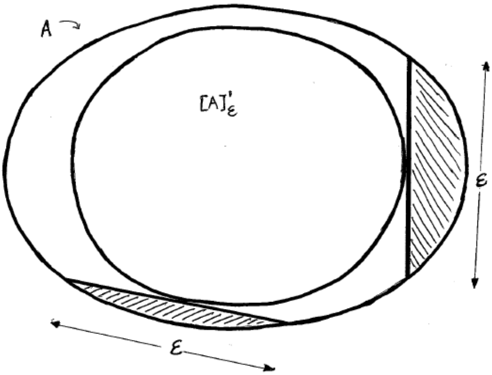

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Closed convex finitely dentable subsets in Banach spaces enjoy many good properties.

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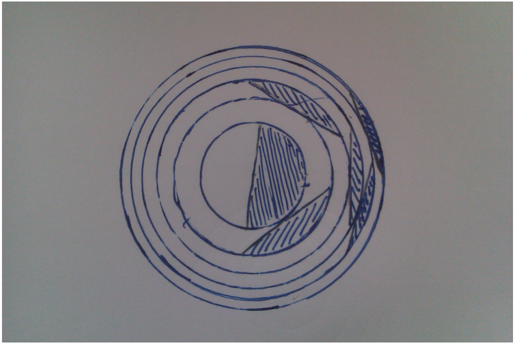

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If $|||\cdot|||$ is another equivalent norm, then

$$\delta_{|||\cdot|||}(\varepsilon) \leq Dz(B_{|||\cdot|||},\varepsilon)^{-1} \leq Dz(B_X, c^2\varepsilon)^{-1}$$

where $c > 1$ is the equivalence constant between the norms.

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A key trick is to change the index $Dz(B_X, t)$ by $\mathfrak{N}_X(t)$.

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- to show that it is under any other upper bound.

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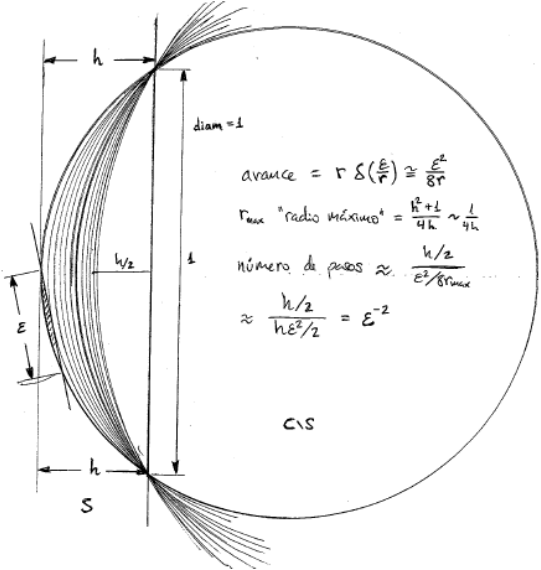

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A small variation can be done above in order to get the submultiplicativity. The function obtained in that fashion is $\mathfrak{N}_X(\varepsilon)$. Anyway, $\mathfrak{N}_X(\varepsilon)$ and $\mathfrak{M}_X(\varepsilon)$ are equivalent functions.

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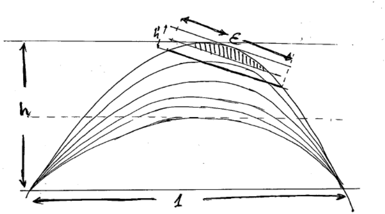

Thank you for your attention!