Sobolev type embeddings into mixed norm spaces

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Rearrangement-invariant Banach function spaces

Definition

A rearrangement invariant Banach function space (briefly an r.i. space) is defined as

$$X(\mathtt{I}^n) = \left\{ f \in \mathcal{M}(\mathtt{I}^n) : \|f\|_{X(\mathtt{I}^n)} < \infty \right\},$$

where $\|\cdot\|_{X(\mathtt{I}^n)}$ satisfies the following properties:

* $\|\cdot\|_{X(\mathtt{I}^n)}$ is a norm;
* if $0 \leq g \leq f$ a.e., then $\|g\|_{X(\mathtt{I}^n)} \leq \|f\|_{X(\mathtt{I}^n)}$;
* if $0 \leq f_j \uparrow f$ a.e., then $\|f_j\|_{X(\mathtt{I}^n)} \uparrow \|f\|_{X(\mathtt{I}^n)}$;
* $\|\chi_{\mathtt{I}^n}\|_{X(\mathtt{I}^n)} < \infty$;
* $\int_{\mathtt{I}^n} |f(x)| dx \lesssim \|f\|_{X(\mathtt{I}^n)}$;
* if $f^* = g^*$, then $\|f\|_{X(\mathtt{I}^n)} = \|g\|_{X(\mathtt{I}^n)}$.

Rearrangement-invariant Banach function spaces

Definition

The Lorentz space $L^{p,q}(\mathtt{I}^n)$, with $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = \infty$, is the space of those $f \in \mathcal{M}(\mathtt{I}^n)$ such that

$$\|f\|_{L^{p,q}(\mathtt{I}^n)} = \left\| t^{1/p-1/q} f^*(t) \right\|_{L^q(\mathtt{I}^n)} < \infty.$$

Definition

Let $0 < p\,, q \leq \infty$ and $\alpha \in \mathbb{R}$. The Lorentz-Zygmund space $L^{p,q;\alpha}(\mathtt{I}^n)$ is the space of all $f \in \mathcal{M}(\mathtt{I}^n)$ for which the expression

$$\|f\|_{L^{p,q;\alpha}(\mathtt{I}^n)} = \left\| t^{1/p-1/q} \left[\log(e/t) \right]^{\alpha} f^*(t) \right\|_{L^q(0,1)}$$

is finite.

Benedek-Panzone spaces

Let $n \in \mathbb{N}$, $n \geq 2$ and $k \in \{1, \dots, n\}$. For any $x \in \mathtt{I}^n$, we denote

$$\widehat{x_k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathtt{I}^{n-1}.$$

Definition

Let $X(\mathtt{I}^{n-1})$ and $Y(\mathtt{I})$ be r.i. spaces. The Benedek-Panzone spaces are defined as

$$\mathcal{R}_k(X, Y) = \left\{ f \in \mathcal{M}(\mathtt{I}^n) : \|f\|_{\mathcal{R}_k(X,Y)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{R}_k(X,Y)} = \|\psi_k(f, Y)\|_{X(\mathtt{I}^{n-1})}, \quad \psi_k(f, Y)(\widehat{x_k}) = \|f(\widehat{x_k}, \cdot)\|_{Y(\mathtt{I})}.$$

Benedek-Panzone spaces

Examples:

The Lebesgue spaces $L^1(\mathbb{I}^n) = \mathcal{R}_1(L^1, L^1)$;

$$\|f\|_{\mathcal{R}_1(L^1,L^1)} = \int_{\mathbb{I}^{n-1}} \psi_1(f, L^1)(\widehat{x_1}) d\widehat{x_1} = \int_{\mathbb{I}^{n-1}} \int_{\mathbb{I}} |f(\widehat{x_1}, x_1)| dx_1 \, d\widehat{x_1}.$$

The Benedek-Panzone spaces $\mathcal{R}_n(L^1, L^2)$;

$$\|f\|_{\mathcal{R}_n(L^1,L^2)} = \int_{\mathbb{I}^{n-1}} \psi_n(f, L^2)(\widehat{x_n}) d\widehat{x_n} = \int_{\mathbb{I}^{n-1}} \left(\int_{\mathbb{I}} |f(\widehat{x_n}, x_n)|^2 dx_n \right)^{1/2} d\widehat{x_n}.$$

Properties:

- $\mathcal{R}_k(X; Y)$ are Banach function spaces;
- Boccuto, Bukhvalov, and Sambucini:

$$\mathcal{R}_k(X, Y) \text{ are r.i. spaces } \iff X = Y = L^p.$$

Mixed norm spaces

Definition

Let $X(\mathtt{I}^{n-1})$ and $Y(\mathtt{I})$ be r.i. spaces. The mixed norm spaces $\mathcal{R}(X, Y)$ are defined as follows

$$\mathcal{R}(X, Y) = \bigcap_{k=1}^{n} \mathcal{R}_k(X, Y).$$

For each $f \in \mathcal{R}(X, Y)$, we set $\|f\|_{\mathcal{R}(X,Y)} = \sum_{k=1}^{n} \|f\|_{\mathcal{R}_k(X,Y)}$.

Examples:

The Lebesgue spaces $L^p(\mathtt{I}^n) = \mathcal{R}(L^p, L^p)$, $1 \leq p \leq \infty$.

Properties:

* $\mathcal{R}(X; Y)$ are Banach function spaces;
* $\mathcal{R}(X, L^\infty)$ are r.i. spaces $\Longleftrightarrow X(\mathtt{I}^{n-1}) = L^\infty(\mathtt{I}^{n-1})$.

Sobolev spaces

We denote $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$, where $\partial_{x_i} u$ is the distributional partial derivate of u with respect to x_i.

Definition

Let $Z(\mathtt{I}^n)$ be an r.i. space. The first-order Sobolev spaces are defined as

$$W^1 Z(\mathtt{I}^n) := \{u \in Z(\mathtt{I}^n) : |\nabla u| \in Z(\mathtt{I}^n)\},$$

with the norm $\|u\|_{W^1 Z(\mathtt{I}^n)} = \|u\|_{Z(\mathtt{I}^n)} + \||\nabla u|\|_{Z(\mathtt{I}^n)}$.

By $W^1_0 Z(\mathtt{I}^n)$ we denote the closure of $C^\infty_c(\mathtt{I}^n)$ in $W^1 Z(\mathtt{I}^n)$.

Classical Sobolev embeddings

Classical Sobolev embedding theorem

$$W^1L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathtt{I}^n), \quad 1 \le p < n.$$

Sobolev, case $p > 1$.

His proof did not apply to $p = 1$.

Gagliardo; Nirenberg, $p = 1$.

$$W^1L^1(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n'}(\mathtt{I}^n).$$

Fournier embedding theorem

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(\mathtt{I}^n).$$

$$W^1L^1(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(\mathtt{I}^n) \hookrightarrow L^{n'}(\mathtt{I}^n).$$

Sobolev embeddings in r.i. spaces

Kerman and Pick studied the Sobolev embeddings among r.i. spaces. In particular, they solved the following problems:

* Given an r.i. range space $X(\mathtt{I}^n)$, find the largest r.i. domain space, namely $Z(\mathtt{I}^n)$, satisfying

$$W^1 Z(\mathtt{I}^n) \hookrightarrow X(\mathtt{I}^n).$$

This means that if $W^1 \widetilde{Z}(\mathtt{I}^n) \hookrightarrow X(\mathtt{I}^n) \Rightarrow \widetilde{Z}(\mathtt{I}^n) \hookrightarrow Z(\mathtt{I}^n)$.

* Given an r.i. domain space $Z(\mathtt{I}^n)$, describe the smallest r.i. range space, namely $X(\mathtt{I}^n)$, that verifies

$$W^1 Z(\mathtt{I}^n) \hookrightarrow X(\mathtt{I}^n).$$

That is, if $W^1 Z(\mathtt{I}^n) \hookrightarrow \widetilde{X}(\mathtt{I}^n) \Rightarrow X(\mathtt{I}^n) \hookrightarrow \widetilde{X}(\mathtt{I}^n)$.

Examples

Classical Sobolev embedding theorem

$W^1L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathtt{I}^n), \quad 1 \leq p < n.$

↓

Hunt; O'Neil; Peetre.

$W^1L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p),p}(\mathtt{I}^n).$

↙ ↘

Kerman and Pick

Optimal r.i. domain space: $L^p(\mathtt{I}^n)$.

Kerman and Pick

Optimal r.i. range space: $L^{pn/(n-p),p}(\mathtt{I}^n)$.

Examples

Critical Sobolev embedding theorem

$$W^1L^n(\mathrm{I}^n) \hookrightarrow L^p(\mathrm{I}^n),\ 1 \leq p < \infty.$$

↓

Maz'ya; Hansson; Brézis and Wainger

$$W^1L^n(\mathrm{I}^n) \hookrightarrow L^{\infty,n;-1}(\mathrm{I}^n).$$

↙ ↘

Kerman and Pick

Optimal r.i. domain space: $Z_{L^{\infty,n;-1}}(\mathrm{I}^n)$,

$$\|f\|_{Z_{L^{\infty,n;-1}}(\mathrm{I}^n)} \approx \left\| \int_t^1 f^*(s) s^{-1/n'} \, ds \right\|_{\overline{X}(0,1)}$$

Hansson; Kerman and Pick

Optimal r.i. range space: $L^{\infty,n;-1}(\mathrm{I}^n)$.

Motivation

Classical Sobolev embeddings

Gagliardo; Nirenberg
$W^1L^1(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^1, L^\infty)$.

Kerman and Pick
Optimal domain and range of
$W^1Z(\mathtt{I}^n) \hookrightarrow Y(\mathtt{I}^n)$,
within the class of r.i. spaces.

Our goal is to describe the largest domain space and the smallest range with mixed norm in $W^1Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty)$.

Problems

Our aim is to study the Sobolev embedding of the form

$$W^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty). \tag{1}$$

In particular, we are interested in the following problems:

* We would like to find the smallest space of the form $\mathcal{R}(X, L^\infty)$ in (1), for a given r.i. $Z(\mathtt{I}^n)$.
* Given a fixed range space $\mathcal{R}(X, L^\infty)$, we would like to describe the largest r.i. domain space satisfying (1).

Problem

Given two r.i. spaces $X(\mathtt{I}^{n-1})$ and $Z(\mathtt{I}^{n})$, give necessary and sufficient conditions for the following embedding to hold

$$W^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty).$$

Theorem

Let $X(\mathtt{I}^{n-1})$ and $Z(\mathtt{I}^{n})$ be r.i. spaces. Then, TFAE:

(i) $W^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty)$;

(ii) $\left\| \int_{t^{n'}}^{1} f^*(s) s^{-1/n'} \, ds \right\|_{\overline{X}(0,1)} \lesssim \|f^*\|_{\overline{Z}(0,1)}, \quad f \in Z(\mathtt{I}^n)$;

(iii) $\|f^{**}(t^{1/n'})\|_{\overline{Z}'(0,1)} \lesssim \|f^*\|_{\overline{X}'(0,1)}, \quad f \in X'(\mathtt{I}^{n-1})$.

Problem

For a fixed r.i. space $Z(\mathtt{I}^n)$, find the smallest of the $\mathcal{R}(X, L^\infty)$, namely $\mathcal{R}(X_{W^1Z,L^\infty}, L^\infty)$, satisfying
$$W^1Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X_{W^1Z,L^\infty}, L^\infty).$$

Theorem

Let $Z(\mathtt{I}^n)$ be an r.i. space and let $X_{W^1Z,L^\infty}(\mathtt{I}^{n-1})$ be the r.i. space whose associate space $X'_{W^1Z,L^\infty}(\mathtt{I}^{n-1})$ has norm given by

$$\|f\|_{X'_{W^1Z,L^\infty}(\mathtt{I}^{n-1})} = \left\|f^{**}(t^{1/n'})\right\|_{\overline{Z}'(0,1)}, \quad f \in \mathcal{M}(\mathtt{I}^{n-1}).$$

Then,

$$W^1Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X_{W^1Z,L^\infty}, L^\infty). \tag{2}$$

Moreover, $\mathcal{R}(X_{W^1Z,L^\infty}, L^\infty)$ is the smallest mixed norm space for (2).

Problem

For a mixed norm space $\mathcal{R}(X, L^\infty)$, find the largest r.i. space $Z(\mathtt{I}^n)$ in

$$W^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty).$$

Theorem

Let $X(\mathtt{I}^{n-1})$ be an r.i. space, with $\overline{\alpha}_X < 1$. Then, the r.i. space $Z(\mathtt{I}^n)$, with norm given by

$$\|f\|_{Z(\mathtt{I}^n)} \approx \Big\| \int_{t^{n'}}^1 f^*(s) s^{-1/n'} ds \Big\|_{\overline{X}(0,1)}, \quad f \in \mathcal{M}(\mathtt{I}^n)$$

satisfies

$$W^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty).$$

Moreover, it is the largest r.i. space for which this embedding holds.

Examples

$$W^1L^p(\mathrm{I}^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty), \quad 1 \le p < n$$

$L^p(\mathrm{I}^n)$ optimal r.i. domain space.

$\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty)$ optimal range of the form $\mathcal{R}(X, L^\infty)$.

$$W^1L^n(\mathrm{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

$Z_{L^{\infty,n;-1}}(\mathrm{I}^n)$ optimal r.i. domain space.

$\mathcal{R}(L^{\infty,n;-1}, L^\infty)$ optimal range of the form $\mathcal{R}(X, L^\infty)$.

Comparison with the optimal r.i. space

$Z(\mathtt{I}^n)$ fixed r.i. domain space.

Kerman and Pick
$X^{\text{op}}(\mathtt{I}^n)$ smallest r.i. range space for
$W^1Z(\mathtt{I}^n) \hookrightarrow X^{\text{op}}(\mathtt{I}^n)$.

$\mathcal{R}(Y, L^\infty)$ smallest range
of the form $\mathcal{R}(X, L^\infty)$ for
$W^1Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(Y, L^\infty)$.

We show that the following chain of embeddings holds
$W^1Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(Y, L^\infty) \hookrightarrow X^{\text{op}}(\mathtt{I}^n)$.

Comparison with the optimal r.i. space

Theorem

Let $Z(\mathtt{I}^n)$ be an r.i. space, let $X^{\mathrm{op}}(\mathtt{I}^n)$ be the smallest r.i. range space satisfying

$$W^1 Z(\mathtt{I}^n) \hookrightarrow X^{\mathrm{op}}(\mathtt{I}^n),$$

and let $\mathcal{R}(Y, L^\infty)$ be the smallest space of the form $\mathcal{R}(X, L^\infty)$ such that

$$W^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(Y, L^\infty).$$

Then,

$$W^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(Y, L^\infty) \hookrightarrow X^{\mathrm{op}}(\mathtt{I}^n).$$

Moreover, $X^{\mathrm{op}}(\mathtt{I}^n)$ is the smallest r.i. space that verifies

$$\mathcal{R}(Y, L^\infty) \hookrightarrow X^{\mathrm{op}}(\mathtt{I}^n).$$

Examples

Domain space: $L^p(\mathtt{I}^n)$, $1 \leq p < n$

Kerman and Pick

$L^{pn/(n-p),p}(\mathtt{I}^n)$ smallest r.i. space for $W^1L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p),p}(\mathtt{I}^n)$.

$\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty)$ smallest space of the form $\mathcal{R}(X, L^\infty)$ satisfying $W^1L^p(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty)$.

$$W^1L^p(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty) \underset{\neq}{\hookrightarrow} L^{pn/(n-p),p}(\mathtt{I}^n).$$

Examples

Domain space: $L^n(\mathrm{I}^n)$

Kerman and Pick
$L^{\infty,n;-1}(\mathrm{I}^n)$ smallest r.i. space for $W^1L^n(\mathrm{I}^n) \hookrightarrow L^{\infty,n;-1}(\mathrm{I}^n)$.

$\mathcal{R}(L^{\infty;n;-1}, L^\infty)$ smallest space of the form $\mathcal{R}(X, L^\infty)$ such that
$$W^1L^n(\mathrm{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

$$W^1L^n(\mathrm{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(\mathrm{I}^n).$$

Critical case of the classical Sobolev embedding

Bastero, Milman and Ruiz, and independently Malý and Pick introduced a new "space".

Definition

Let $1 \leq q < \infty$. The space $L(\infty, q)(\mathtt{I}^n)$ is the collection of all $f \in \mathcal{M}(\mathtt{I}^n)$ such that

$$\|f\|_{L(\infty,q)(\mathtt{I}^n)} = \left(\int_0^1 \left[f^{**}(s) - f^*(s) \right]^q \frac{dt}{t} \right)^{1/q} < \infty.$$

Theorem (Bastero-Milman-Ruiz, Malý-Pick)

* $L(\infty, q)(\mathtt{I}^n)$ *is not a linear set;*
* $L(\infty, n)(\mathtt{I}^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(\mathtt{I}^n)$.

Critical case of the classical Sobolev embedding

Bastero, Milman and Ruiz, and independently Malý and Pick saw that a further improvement of

$$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L^{\infty,n;-1}(\mathtt{I}^n)$$

was possible within the class of non-linear r.i. spaces.

Theorem (Bastero-Milman-Ruiz, Malý-Pick)

$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L(\infty, n)(\mathtt{I}^n)$.

Corollary (Bastero-Milman-Ruiz, Malý-Pick)

$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L(\infty, n)(\mathtt{I}^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(\mathtt{I}^n)$.

Motivation

Critical Sobolev embedding theorem

$$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L^p(\mathtt{I}^n), \quad 1 \leq p < \infty.$$

$\mathcal{R}(L^{\infty;n;-1}, L^\infty)$ smallest space of the form $\mathcal{R}(X, L^\infty)$ such that

$$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Kerman and Pick

$L^{\infty,n;-1}(\mathtt{I}^n)$ smallest r.i. space for $W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L^{\infty,n;-1}(\mathtt{I}^n)$.

Bastero, Milman and Ruiz; Malý and Pick

Improvement among non-linear r.i. spaces

$$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L(\infty, n)(\mathtt{I}^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(\mathtt{I}^n).$$

Try to find an improvement of $W_0^1 L^n(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty)$ among non-linear mixed norm spaces.

Comparison with the optimal mixed norm space

Problem

Are $L(\infty, n)(\mathtt{I}^n)$ and $\mathcal{R}(L^{\infty,n;-1}, L^{\infty})$ comparable?

Theorem

$\mathcal{R}(L^{\infty,n;-1}, L^{\infty}) \not\hookrightarrow L(\infty, n)(\mathtt{I}^n)$.

Theorem

Let $X(\mathtt{I}^{n-1})$ *be an r.i. space. Then,* $L(\infty, n)(\mathtt{I}^n) \not\hookrightarrow \mathcal{R}(X, L^{\infty})$.

Sobolev embedding in non-linear mixed norm spaces

Problem

Try to find an improvement of $W_0^1 L^n(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty)$, within the class of non-linear spaces of the form $\mathcal{R}(X, L^\infty)$.

Theorem

$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty)$.

Proposition

* $\mathcal{R}(L(\infty, n), L^\infty)$ *is not a linear set;*
* $L(\infty, n)(\mathtt{I}^n) \not\hookrightarrow \mathcal{R}(L(\infty, n), L^\infty)$;
* $\mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty)$.

Corollary

$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty)$.

The end

Thank You!!