The Bishop-Phelps-Bollobás property for operators

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Bishop-Phelps Theorem

X (real or complex) Banach space, X^* the dual space

Bishop-Phelps Theorem, 1961

$$\{x^* \in X^* : x^* \text{ attains its norm } \} \quad \text{is dense in } X^*$$

$$B_X = \{x \in X : \|x\| \le 1\}, \quad S_X = \{x \in X : \|x\| = 1\}$$

Bishop-Phelps-Bollobás Theorem, 1970

X Banach, $0 < \varepsilon < 1$. Given $(x, x^*) \in S_X \times S_{X^*}$ with $|1 - x^*(x)| < \dfrac{\varepsilon^2}{4}$

$$\exists (y, y^*) \in S_X \times S_{X^*} : y^*(y) = 1, \|y - x\| < \varepsilon, \|y^* - x^*\| < \varepsilon \, .$$

Some application of Bishop-Phelps-Bollobás Theorem

Second volumen by Bonsall and Duncan devoted to numerical range of operators

X, Y Banach spaces

$$L(X,Y) = \{T : X \longrightarrow Y \ : \ T \text{ is linear and bounded } \}$$

If $T \in L(X,Y)$, the numerical range of T is given by

$$V(T) = \{x^*(T(x)) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

Trivial that $V(T) \subset V(T^*)$.

It is satisfied that

$$V(T^*) = \overline{V(T)}.$$

Norm attaining operators

$$NA(X, Y) = \{T \in L(X, Y) : \exists x \in B_X, \|Tx\| = \|T\|\}$$

Question: Is $NA(X, Y)$ (norm) dense in $L(X, Y)$?

(Lindenstrauss, 1963)

1) For $X = c_0$, Y strictly convex and isomorphic to c_0,

$$\overline{NA(X, Y)} \neq L(X, Y)$$

2) X reflexive $\Rightarrow$ $\overline{NA(X, Y)} = L(X, Y), \ \forall Y.$

3) $\overline{NA(\ell_1, Y)} = L(\ell_1, Y), \ \forall Y.$

4) $\overline{NA(X, \ell_\infty)} = L(X, \ell_\infty), \ \forall X.$

Versions of Bishop-Phelps Theorem for operators

(Bourgain, 1977)

X has the Radon-Nikodým property $\Rightarrow$ $\overline{NA(X,Y)} = L(X,Y), \ \forall Y$.

A few positive results for classical spaces

(Iwanik, 1979)

For every finite positive Borel measures μ, ν on $[0,1]$,
$\overline{NA(L_1(\mu), L_1(\nu))} = L(L_1(\mu), L_1(\nu))$

(Johnson and Wolfe, 1979)

If K and S are compact and Hausdorff topological spaces, then
$\overline{NA(C(K), C(S))} = L(C(K), C(S))$ (real valued functions).

(Schachermayer, 1983) and (Alaminos, Choi, Kim and Payá, 1998)

If L is a Hausdorff and locally compact topological space, then
$\overline{WC(C_0(L), Y) \cap NA(C_0(L), Y)} = WC(C_0(L), Y), \ \ \forall Y$.

WC= weakly compact operators

Versions of Bishop-Phelps Theorem for operators

Hence

$$\overline{NA(C(K), \ell_2)} = L(C(K), \ell_2) \quad \text{and} \quad \overline{NA(C(K), L_1[0,1])} = L(C(K), L_1[0,1]) \, .$$

Some counterexamples

(Schachermayer, 1979), (Johnson and Wolfe, 1979)

$$\overline{NA(L_1[0,1], C[0,1])} \neq L(L_1[0,1], C[0,1])$$

(Gowers, 1990), (Aguirre, 1990), (A, 1998)

$$\exists X \ : \ \overline{NA(X,Y)} \neq L(X,Y)$$

for every Y with $\dim Y = \infty$, Y strictly convex

$$\exists X \ : \ \overline{NA(X, L_1(\mu))} \neq L(X, L_1(\mu))$$

whenever $\dim L_1(\mu) = \infty$.

(Martín, 2014)

$$\exists X, Y \ : \ \overline{NAK(X,Y)} \neq K(X,Y))$$

($K(X,Y)$=compact operators from X into Y)

Bishop-Phelps-Bollobás property for operators

Bishop-Phelps-Bollobás property for operators

(X, Y) has the Bishop-Phelps-Bollobás property (BPBp) for operators if for every $0 < \varepsilon < 1$ there are $\eta(\varepsilon) > 0$ satisfying that

$$T \in S_{L(X,Y)}, \; x_0 \in S_X : \|Tx_0\| > 1 - \eta(\varepsilon) \quad \Rightarrow$$

$$\exists S \in S_{L(X,Y)}, u_0 \in S_X : \quad \|S(u_0)\| = 1,$$

$$\|u_0 - x_0\| < \varepsilon \;\text{ and }\; \|S - T\| < \varepsilon$$

Question: Which pair of spaces (X, Y) have the BPBp?

Some known answers I

(A., Aron, García and Maestre, 2008)

1) $\dim X, \dim Y < +\infty \quad \Rightarrow \quad (X, Y)$ has the BPBp for operators.

2)

$$(\ell_1, Y) \text{ satisfies } \textit{BPBp} \text{ for operators} \quad \Leftrightarrow \quad Y \text{ has the } \textit{AHSp}$$

AHSp= almost hyperplane series property

3) Finite-dimensional spaces, uniformly convex spaces, $C(K)$ and $L_1(\mu)$ (μ σ-finite) satisfy AHSp

Some known answers II

(Aron, Choi, García and Maestre, 2011)

μ σ-finite measure $\Rightarrow$ $(L_1(\mu), L_\infty[0,1])$ has the BPBp for operators.

(Aron, Cascales, Kozhushkina, 2011)

$$\left.\begin{array}{l} X \quad \text{Asplund} \\ \\ L \ \text{ locally compact Hausdorff} \end{array}\right\} \Rightarrow (X, C_0(L)) \text{ has the BPBp for operators.}$$

(A., Becerra, García and Maestre, 2014), (Dai, preprint), (Kim, Lee, 2013)

X is uniformly convex $\Rightarrow$ (X, Y) has the BPBp for operators.

Some known results III

(Kim, 2013)

1)

$$Y \text{ uniformly convex} \quad \Rightarrow \quad (c_0, Y) \text{ has the BPBp for operators}$$

2) Y strictly convex,

$$(c_0, Y) \text{ has the BPBp for operators} \quad \Leftrightarrow \quad Y \text{ is uniformly convex}$$

(A., Becerra, Choi, Ciesielski, Kim, Lee, Lourenço, Martín, 2014)

K, S compact Hausdorff spaces,
$(C(K), C(S))$ has the BPBp for operators (for real valued functions).

Very recent results for operators acting on $C_0(L)$

(A., Becerra, García, Kim, Maestre, preprint, 2013)

1) Characterize the spaces Y such that (ℓ_∞^3, Y) has the BPBp for operators (real case).

2) Finite-dimensional spaces, uniformly convex spaces, $C_0(L)$ (L locally compact Hausdorff space) and $L_1(\mu)$ satisfy the previous condition.

3) If a strictly convex space Y satisfies 1) then it is uniformly convex.

4) Every **real** Banach space admits an equivalent norm satisfying **1)**.

Very recent results for operators acting on $C_0(L)$

(Kim, Lee, 2015)

Y uniformly convex, K compact Hausdorff space
$(C(K), Y)$ has the BPBp for operators (real case).

(Kim, Lee, Lin, preprint, 2014)

prove that
Y uniformly convex, $X = L_\infty(\mu)$ or $X = c_0(\Gamma)$. Then
(X, Y) has the BPBp for operators (real case).

state that
Y is complex uniformly convex, $X = L_\infty(\mu)$ or $X = c_0$ (complex cases). Then
(X, Y) has the BPBp for operators (complex case).

Very recent results for operators acting on $C_0(L)$

Theorem

$(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property for operators for any locally compact Hausdorff topological space L and any $\mathbb{C}$-uniformly convex space Y.
Moreover the function η (Definition of BPBp) depends only on the modulus of convexity of Y.

Definition

Y complex space, the $\mathbb{C}$*-modulus of convexity* δ is given by

$$\delta(\varepsilon) = \inf\big\{\sup\{\|x + \lambda\varepsilon y\| - 1 : \lambda \in \mathbb{C}, |\lambda| = 1\big\} : x, y \in S_Y\big\}.$$

Y is $\mathbb{C}$*-uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$.

Examples:
1) Every uniformly convex complex space is $\mathbb{C}$-uniformly convex and the converse is not true.
2)The complex space $L_1(\mu)$ is $\mathbb{C}$-uniformly convex (Globevnik).

Facts used in the proof

Proposition

Let Y be a $\mathbb{C}$-uniformly convex Banach space and L any locally compact Hausdorff space. Then every operator from $C_0(L)$ into Y is weakly compact.

Argument in the proof use James distortion Theorem + Bessaga-Pełczyński selection principle.

(If X is a complex space and $X_{\mathbb{R}}$ contains c_0 (real) then X contains also the complex c_0.)

Facts used in the proof

L a locally compact Hausdorff space,
$\mathcal{B}(L)$ = Borel measurable and bounded complex valued functions on L (with the sup norm)
$B \subset L$ Borel measurable set, define $P_B : \mathcal{B}(L) \longrightarrow \mathcal{B}(L)$

$$P_B(f) = f\chi_B \quad (f \in \mathcal{B}(L)).$$

By Riesz Theorem, the space $\mathcal{B}(L)$ is embedded as a subspace of $C_0(L)^{**}$ such that the restriction of this identification to $C_0(L)$ is the usual injection of a space to its bidual. As a consequence, for an operator $T \in L(C_0(L), Y)$ and a Borel set $B \subset L$, the composition $T^{**}P_B$ makes sense.

Lemma

Let Y be a $\mathbb{C}$-uniformly convex space with modulus of $\mathbb{C}$-convexity δ.
Let B a Borel set of L. Assume that for some $0 < \varepsilon < 1$ and $T \in S_{L(C_0(L),Y)}$ it is satisfied $\|T^{**}P_B\| > 1 - \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}$. Then $\|T^{**}(I - P_B)\| \leq \varepsilon$.

Some ideas and results used in the proof I

1) Riesz Theorem, Radon-Nikodým Theorem

2) If $\mu \in M(K) = C(K)^*$, $\|\mu\| = 1$, $g = \frac{d\mu}{d|\mu|}$.
Then μ attains its norm at $\overline{g}$.

Some estimates of the same kind
Assume $\mu \in M(K) = C(K)^*$, $\|\mu\| = 1$, $g = \frac{d\mu}{d|\mu|}$. Assume $|g| = 1$.
If $f \in S_{C(K)}$ satisfies that $\mathrm{Re}\, \mu(f) = \mathrm{Re}\, \int_K fgd|\mu| > 1 - \varepsilon^2$. Take

$$A = \{t \in K : \mathrm{Re}\, f(t)g(t) > 1 - \varepsilon\}.$$

Then $|\mu|(K \backslash A) < \varepsilon$ and $\|(f - \overline{g})\chi_A\|_\infty \leq \sqrt{2\varepsilon}$.

Some ideas and results used in the proof II

3) Lusin Theorem

4) If Y is $\mathbb{C}$-uniformly convex then Y does not contain c_0. Hence any operator from $C(K)$ into Y is weakly compact (Pełczyński).

5) The set of weakly compact norm attaining operators is dense in the set of weakly compact operators from $C(K)$ into Y (Alaminos, Choi, Kim, Payá)

6) Urysohn Lemma

Sketch of the proof I

Sketch of proof:
1) Given $0 < \varepsilon < 1$, take $\eta = \eta(\varepsilon, \delta(\varepsilon))$ and $s = s(\varepsilon)$. Assume that $T : C_0(L) \longrightarrow Y$ and f_0 satisfies that

$$T \in S_{L(C_0(L),Y)}, \qquad f_0 \in S_{C_0(L)}, \qquad \text{and} \qquad \|Tf_0\| > 1 - s.$$

Write $\mu_1 = T^* y_1^* \in B_{C_0(L)^*} \equiv B_{M(L)}$ (Riesz Theorem) and $g_1 = \dfrac{d\mu_1}{|\mu_1|}$. We can assume that $|g_1| = 1$. A by

$$A = \{t \in L : \mathrm{Re}\, f_0(t) g_1(t) > 1 - \beta\}.$$

Immediate to obtain that

$$\|(f_0 - \overline{g_1})\chi_A\|_\infty \leq \sqrt{2\beta} = \frac{\varepsilon}{12}.$$

Prove that

$$|\mu_1|(L \backslash A) \leq \frac{s}{\beta} \sim 0$$

Sketch of the proof II

2) By Lusin Theorem we approximate A by a compact set.
There is a compact set $B \subset A$ such that $g_{1|B}$ is continuous and $|\mu_1|(A \backslash B) \leq \frac{\varepsilon\eta}{2}$ (almost zero), so $|\mu_1|(L \backslash B)$ is also close to zero.
Hence $\|T^{**}P_B\| \geq |\mu_1| > 1 - \eta$. By Lemma we obtain that $\|T^{**}(I - P_B)\| \leq \frac{\varepsilon}{9}$.
3) By using y_1^* and g_1 we define an operator $\tilde{S}$ close to T and such that

$$\tilde{S}^{**} = \tilde{S}^{**}P_B \qquad \text{and} \qquad \|\tilde{S}\| \sim 1.$$

Let S_1 be the restriction of $\tilde{S}$ to $C(B)$ ($S_1(f) = \tilde{S}^{**}(f\chi_B)$). Use that in this case $NA(C(B), Y)$ is dense in $L(C(B), Y)$ (Alaminos, Choi, Kim, Payá).
There is an operator $S_2 \in L(C(B), Y)$ and $h_1 \in S_{C(B)}$ satisfying that

$$\|\tilde{S}\| = \|S_2\| = \|S_2(h_1)\| \qquad \text{and} \qquad \|S_2 - S_1\| < \frac{\varepsilon\eta}{2}.$$

We can choose $y_2^* \in S_{Y^*}$ such that $y_2^*\big(S_2(h_1)\big) = \|S_2\|$ and check that

$$\mathrm{Re}\; y_2^*\big(R_2(\overline{g_1}_{|B})\big) \sim 1,$$

where

$$R_2 = \frac{S_2}{\|S_2\|}, \qquad \mu_2 = R_2^*(y_2^*) \in M(B), \qquad \text{and} \qquad g_2 = \frac{d\mu_2}{d|\mu_2|}, \quad |g_2| = 1.$$

Sketch of the proof III

Consider the subset C given by

$$C = \left\{ t \in B : \mathrm{Re}\, \left(\overline{g_1}(t) + h_1(t)\right) g_2(t) > 2 - \beta \right\}.$$

We check that $|\mu_2|(B \backslash C) \sim 0$ and we also have that

$$\|(h_1 - f_0)\chi_C\| \leq \frac{\varepsilon}{4}.$$

By the inner regularity of μ_2 there is a compact ser $K_1 \subset C$ such that $|\mu_2|(C \backslash K_1) \sim 0$ and also $\|R_2^{**} P_{K_1}\| \sim 1$. By the Lemma we obtain $\|\|\|R_2^{**}(P_B - P_{K_1})\| \sim 0$. We also have that $K_1 \neq \varnothing$.

Sketch of the proof IV

5) By Urysohn Lemma and up to some modifications we can assume that the function h_1 satisfies that $|h_1(t_0)| = 1$ for some $t_0 \in K_1$. Since $B \subset L$ is compact and $\|h_1\| = 1$ there is a function $f_2 \in S_{C_0(L)}$ that extends h_1.
We knew that $(h_1 - f_0)\chi_{K_1}$ is close to zero, so there is an open set G such that $K_1 \subset G$ and $(f_2 - f_0)\chi_G$ is close to zero.
In view of Urysohn Lemma there is a function $f_3 \in S_{C_0(L)}$ such that

$$\|f_3 - f_0\| < \varepsilon \qquad \text{and} \qquad f_{3|K_1} = h_1.$$

6) Define S by

$$S(f) = R_2^{**}\big((f\chi_{K_1})_{|B}\big) + \lambda_0 f(t_0) R_2^{**}\big(h_1\chi_{B\setminus K_1}\big)$$

for some λ_0 satisfying $|\lambda_0| = 1$.
Since R_2 is weakly compact, $S \in L(C_0(L), Y)$.

$$\|R_2\| = 1 \qquad \Rightarrow \qquad \|S\| \leq 1.$$

Also it is satisfied that $\|S - T\| < \varepsilon$ and $\|f_3 - f_0\| < \varepsilon$.

Open problems

Some open questions

- Does the pair (c_0, ℓ_1) (real case) have the BPBp for operators?
- Characterize the Banach spaces Y such that the pair $(L_1(\mu), Y)$ satisfies the BPBp for operators.