On extreme operators whose adjoints preserve extreme points

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$$E_X = S_X \setminus \{\tfrac{x+y}{2} : x, y \in S_X, x \neq y\}$$

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$$E(X, Y) = E_{\mathcal{L}(X,Y)}$$

Definitions

Definition (Nice operator)

Let X and Y be Banach spaces. An operator T in $\mathcal{L}(X, Y)$ is said to be *nice* if $T^*(E_{Y^*}) \subseteq E_{X^*}$.

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Definition (Nice space)

A Banach space X is said to be *nice* if $N(Y, X) = E(Y, X)$ for every Banach space Y.

Proposition

Let X be a Banach space, then there exists a Banach space Y such that

$$N(X, Y) \neq E(X, Y).$$

Proposition

Let I be a nonempty set, then $c_0(I)$ is nice.

The results

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Theorem

Let X be a Banach space such that there exists $e_0^* \in E_{X^*}$ which satisfies:

i) $X^* = \overline{\mathrm{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$.

ii) For each $e^* \in E_{X^*} \setminus \{\pm e_0^*\}$, there exists $x \in B_X$ such that $e_0^*(x) = 0$ and $e^*(x) = 1$.

Then X is non-nice.

Consequences

Corollary

Let L be a locally compact Hausdorff space, and let L' denote the set of all cluster points of L. If L' is nonempty, then $\mathcal{C}_0(L)$ is non-nice.

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Corollary

Let L be a locally compact Hausdorff space, then $\mathcal{C}_0(L)$ is nice if and only if L is discrete. In particular, if K is a compact Hausdorff space, then $\mathcal{C}(K)$ is nice if and only if K is finite.

Consequences

Corollary

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that μ is σ-finite, and set $L_1(\mu) := L_1(\Omega, \mathcal{A}, \mu)$. If $\dim(L_1(\mu)) > 2$, then $L_1(\mu)$ is non-nice.

Consequences

Corollary

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that μ is σ-finite, and set $L_1(\mu) := L_1(\Omega, \mathcal{A}, \mu)$. If $\dim(L_1(\mu)) > 2$, then $L_1(\mu)$ is non-nice.

If $\dim(L_1(\mu)) = 1$ or 2, we have that $L_1(\mu)$ is isometrically isomorphic, respectively, to $\mathbb{R}$ and to l_∞^2, so the space is nice.

The results

Proposition

Let X be a Banach space that is nice and such that E_X is nonempty. Then

$$|x^*(x)| = 1 \text{ for all } x^* \in E_{X^*} \text{ and } x \in E_X.$$

Finite-dimensional spaces

Definition

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Proposition (Lindenstrauss and Perles)

If X is a finite-dimensional Banach space, then the following assertions are equivalent:

i) $N(X,X) = E(X,X)$.

ii) $T \in E(X,X),\ x \in E_X \Rightarrow T(x) \in E_X$.

Finite-dimensional spaces

Theorem

Let X be a finite-dimensional Banach space, then X is nice if, and only if, $X = l_\infty^n$ for some $n \in \mathbb{N}$.

References

A. M. Cabrera-Serrano and J. F. Mena-Jurado. On extreme points whose adjoints preserve extreme points, *J. Convex Anal.* **22** (2015) (To appear).