Compact bilinear commutators: the weighted case

Wendolín Damián
Universidad de Sevilla

Joint work with
Á. Bényi, K. Moen and R.H. Torres

Outline

Bilinear Calderón–Zygmund operators

DEFINITION (GRAFAKOS-TORRES, 2002)

We say that T is an bilinear **Calderón-Zygmund operator** if, for some $1 < p_1, p_2 < \infty$, it extends to a bounded bilinear operator from $L^{p_1} \times L^{p_2}$ to L^p, where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},$$

and if there exists a function K, defined off the diagonal $x = y = z$ in $(\mathbb{R}^n)^3$, satisfying

$$T(f,g)(x) = \iint_{\mathbb{R}^{2n}} K(x,y,z) f(y) g(z)\, dydz,$$

for all $x \notin supp f \cap supp\, g$.

Bilinear Calderón–Zygmund operators

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The kernel K must also satisfy these **conditions**:

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and

$$|\nabla K(x,y,z)| \lesssim \frac{1}{\Big(|x-y|+|x-z|\Big)^{2n+1}},$$

where ∇ denotes the gradient in all possible variables.

Class of multiplicative symbols

BOUNDED MEAN OSCILLATION FUNCTIONS

$$BMO = \{ f \in L^1_{loc}(\mathbb{R}^n) : \; M^\sharp f \in L^\infty \}$$

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FEFFERMAN-STEIN SHARP MAXIMAL FUNCTION

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy$$

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$$||f||_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx.$$

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CONTINUOUS MEAN OSCILLATION FUNCTIONS

CMO is defined as the closure of C_c^∞ in the *BMO* norm.

Commutators of linear Calderón–Zygmund operators

Let T be a linear Calderón–Zygmund operator associated with a kernel K and b is a BMO function.

$$\begin{aligned}[T,b]f(x) &= T(bf)(x) - bT(f)(x) \\ &= \int_{\mathbb{R}^n} (b(y)-b(x))K(x,y)f(y)dy.\end{aligned}$$

Commutators of bilinear CZO

Let T be a bilinear Calderón–Zygmund operator and $b \in BMO$.

COMMUTATOR IN THE FIRST VARIABLE

$$\begin{aligned}[T,b]_1(f,g) &= T(bf,g)(x) - bT(f,g)(x) \\ &= \iint_{(\mathbb{R}^n)^2} K(x,y,z)(b(y)-b(x))f(y)g(z)dydz.\end{aligned}$$

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COMMUTATOR IN THE SECOND VARIABLE

$$\begin{aligned}[T,b]_2(f,g) &= T(f,bg)(x) - bT(f,g)(x)\\ &= \iint_{(\mathbb{R}^n)^2} K(x,y,z)(b(z)-b(x))f(y)g(z)dydz.\end{aligned}$$

Iterated commutators of bilinear CZO

Let T be a bilinear Calderón–Zygmund operator and $\vec{b} = (b_1, b_2) \in BMO^2$.

$$[T, \vec{b}](f,g)(x) = [[T, b_1]_1, b_2]_2(f,g)(x) = [[T, b_2]_2, b_1]_1(f,g)(x)$$

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For a multi-index $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$,

$$[T,\vec{b}]_{\vec{\alpha}}(f,g)(x) = \iint_{(\mathbb{R}^n)^2} K(x,y,z)(b_1(y)-b_1(x))^{\alpha_1}((b_2(z)-b_2(x))^{\alpha_2}f(y)g(z)dydz.$$

Compactness of a bilinear operator

Let X, Y and Z be normed spaces and $T : X \times Y \to Z$ be a bilinear operator.

Definition (Bényi and Torres, 2013)

1. Jointly compact if $\{T(x,y) : \|x\|_X, \|y\|_Y \leq 1\}$ is precompact in Z.

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4. Separately compact if T is compact both in the first and second variable.

Boundedness of linear commutators

In 1976, Coifman, Rochberg and Weiss proved:

THEOREM

Let T be a Calderón–Zygmund operator. If $b \in BMO$, then

$$[T,b] : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 1 < p < \infty.$$

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Conversely, if for every $j = 1, \ldots, n$, $[R_j, b]$ is bounded on $L^p(\mathbb{R}^n)$ for some p, $1 < p < \infty$, where R_j is the j-th Riesz transform given by

$$R_j f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \, p.v. \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y)dy, \qquad 1 \leq j \leq n,$$

then $b \in BMO$.

Compactness of linear commutators

Theorem (Uchiyama, 1978)

Let $b \in \cup_{q>1} L^q_{loc}(\mathbb{R}^n)$ and T a Calderón–Zygmund operator. Then $[T, b]$ is a compact operator from $L^p(\mathbb{R}^n)$ into itself, $1 < p < \infty$, if and only if, $b \in CMO$.

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Theorem (Fréchet-Kolmogorov)

A set $\mathscr{K}$ is precompact in L^p, $1 \leq p < \infty$, if and only if

1. $\mathscr{K}$ is bounded in L^p;
2. $\lim_{A\to\infty} \int_{|x|>A} |f(x)|^p \, dx = 0$ uniformly for $f \in \mathscr{K}$;
3. $\lim_{t\to 0} \|f(\cdot + t) - f\|_{L^p} = 0$ uniformly for $f \in \mathscr{K}$.

Boundedness and compactness of bilinear commutators

THEOREM

Let T be a bilinear Calderón–Zygmund operator and $b, b_1, b_2 \in BMO$. Then $[T,b]_1$, $[T,b]_2$ and $[[T,b_1]_1,b_2]_2 : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 < p_1, p_2 < \infty$, with estimates of the form

$$||[T,b]_1(f,g),[T,b]_2(f,g)||_{L^p} \lesssim ||b||_{BMO}||f||_{L^{p_1}}||g||_{L^{p_2}},$$

$$||[[T,b_1]_1,b_2]_2(f,g)||_{L^p} \lesssim ||b_1||_{BMO}||b_2||_{BMO}||f||_{L^{p_1}}||g||_{L^{p_2}}.$$

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$$||[[T,b_1]_1,b_2]_2(f,g)||_{L^p} \lesssim ||b_1||_{BMO}||b_2||_{BMO}||f||_{L^{p_1}}||g||_{L^{p_2}}.$$

THEOREM (BÉNYI AND TORRES, 2013)

If $b, b_1, b_2 \in CMO$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 < p_1, p_2 < \infty$ and $1 \leq p < \infty$, then $[T,b]_1$, $[T,b]_2$ and $[[T,b_1]_1,b_2]_2$ are compact for the same range of exponents.

A_p weights

In 1972, Muckenhoupt characterized the class of weights v for which the following strong inequality holds

$$\int_{\mathbb{R}^n} (Mf(x))^p v(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad f \in L^p(v),$$

where

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

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holds if and only if v satisfies the A_p condition

A_p CONDITION

$$[v]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q v(x)dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p-1}} \right)^{p-1} < \infty, \quad p > 1.$$

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Theorem (Coifman and C. Fefferman, 1974)

Let T be a Calderón–Zygmund operator. Then, for any $w \in A_p$, $1 < p < \infty$,

$$T : L^p(w) \to L^p(w).$$

Compactness in the weighted setting

THEOREM (CLOP AND CRUZ, 2013)

Let $1 < p < \infty$ and $w \in A_p$ and let $\mathscr{K} \subset L^p(w)$. If

1. $\mathscr{K}$ is bounded in $L^p(w)$;
2. $\lim_{A \to \infty} \int_{|x|>A} |f(x)|^p w(x)\,dx = 0$ uniformly for $f \in \mathscr{K}$;
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THEOREM (CLOP AND CRUZ, 2013)

Let T be a Calderón–Zygmund operator. Let $w \in A_p$, with $1 < p < \infty$, and let $b \in CMO$. Then the commutator $[T, b] : L^p(w) \to L^p(w)$ is compact.

$A_{\vec{P}}$ condition

Let $\vec{P} = (p_1, \cdots, p_m)$ and let p be a number such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

DEFINITION (LOPTT, 2009)

Let $1 \leq p_1, \ldots, p_m < \infty$. Given $\vec{w} = (w_1, \ldots, w_m)$, set $\nu_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}$.

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$A_{\vec{P}}$ CONDITION (LOPTT, 2009)

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right) \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q w_j^{1-p_j'} \right)^{p/p_j'} < \infty.$$

When $p_j = 1$, $\left(\frac{1}{|Q|} \int_Q w_j^{1-p_j'} \right)^{p/p_j'}$ is understood as $(\inf_Q w_j)^{-p}$.

Weighted results in the multilinear setting

Theorem (LOPTT, 2009)

Let $1 < p_j < \infty, j = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds for every $\vec{f}$ if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition, where

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i.$$

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holds for every $\vec{f}$ if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition.

THEOREM (LOPTT, 2009)

Let T be a multilinear Calderón–Zygmund operator. If $\vec{w} \in A_{\vec{P}}$ with $\vec{P} > \vec{1}$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, then

$$T : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(\nu_{\vec{w}}).$$

Main results

Theorem I (Bényi, D., Moen and Torres)

Suppose $\vec{P} \in (1,\infty) \times (1,\infty)$, $p = \frac{p_1 p_2}{p_1+p_2} > 1$, $b \in CMO$, and $\vec{w} \in A_p \times A_p$. Then $[T,b]_1$ and $[T,b]_2$ are compact operators from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.

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Theorem II (Bényi, D., Moen and Torres)

Suppose $\vec{P} \in (1,\infty)\times(1,\infty)$, $p = \frac{p_1 p_2}{p_1+p_2} > 1$, $\vec{b} \in CMO \times CMO$, and $\vec{w} \in A_p \times A_p$. Then $[T,\vec{b}]$ is a compact operator from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.

Sketch of proof for $[T, b]_1$

1. Introduce the truncated kernels

$$K^\delta(x,y,z) = \begin{cases} K(x,y,z), & \max(|x-y|, |x-z|) > \delta \\ 0, & \max(|x-y|, |x-z|) \le \delta. \end{cases}$$

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2. Consider the operator $T^\delta(f,g)$ associated with K^δ. We get that

$$|[T^\delta,b]_1(f,g)(x) - [T,b]_1(f,g)(x)| \lesssim \delta \|\nabla b\|_{L^\infty} \mathscr{M}(f,g)(x).$$

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$$|[T^\delta, b]_1(f,g)(x) - [T,b]_1(f,g)(x)| \lesssim \delta \|\nabla b\|_{L^\infty} \mathscr{M}(f,g)(x).$$

3. Since the bounds of the commutators with *BMO* functions are

$$\|[T,\vec{b}]_{\vec{\alpha}}(f,g)\|_{L^p(v_{\vec{w}})} \lesssim \|b_1\|_{BMO}^{\alpha_1} \|b_2\|_{BMO}^{\alpha_2} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)},$$

to show compactness when working with symbols in *CMO* we may also assume $\vec{b} \in C_c^\infty \times C_c^\infty$ by density and the estimates may depend on $\vec{b}$ too.

Sketch of proof for $[T,b]_1$

Fix $\delta > 0$ and assume $b \in C_c^\infty$. Suppose f,g belong to

$$B_{1,L^{p_1}(w_1)} \times B_{1,L^{p_2}(w_2)} = \{(f,g) : \|f\|_{L^{p_1}(w_1)}, \|g\|_{L^{p_2}(w_2)} \leq 1\},$$

with w_1 and w_2 in A_p.

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with w_1 and w_2 in A_p. We want to prove:

1. $[T^\delta, b]_1(B_{1,L^{p_1}(w_1)} \times B_{1,L^{p_2}(w_2)})$ is bounded in $L^p(\nu_{\vec{w}})$;
2. $\lim\limits_{R\to\infty} \int_{|x|>R} |[T^\delta, b]_1(f,g)(x)|^p \nu_{\vec{w}}\, dx = 0;$
3. $\lim\limits_{t\to 0} \|[T^\delta, b]_1(f,g)(\cdot + t) - [T^\delta, b]_1(f,g)\|_{L^p(\nu_{\vec{w}})} = 0.$

Sketch of proof for $[T, b]_1$

Proof of condition 1. It is a consequence of the following

Theorem (Pérez, Pradolini, Torres and Trujillo-González)

Let T be a bilinear Calderón–Zygmund operator and $\vec{w} \in A_{\vec{P}}$ with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},$$

and $1 < p_1, p_2 < \infty$ and $\vec{b} = (b_1, b_2) \in BMO^2$. Then

$$||[[T, b_1]_1, b_2]_2(f, g)||_{L^p(v_{\vec{w}})} \lesssim ||b_1||_{BMO}||b_2||_{BMO}||f||_{L^{p_1}(w_1)}||g||_{L^{p_2}(w_2)}.$$

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$$||[[T, b_1]_1, b_2]_2(f, g)||_{L^p(v_{\vec{w}})} \lesssim ||b_1||_{BMO}||b_2||_{BMO}||f||_{L^{p_1}(w_1)}||g||_{L^{p_2}(w_2)}.$$

Observe that $\vec{w} \in A_p \times A_p \subset A_{\vec{P}}$.

Sketch of proof for $[T,b]_1$

Proof of condition 2. Let A be large enough so that $supp\, b \subset B_A(0)$ and let $R \geq \max(1, 2A)$. Then for $|x| > R$ we have

$$|[T^{\delta}, b]_1(f,g)(x)| \leq \|b\|_{L^\infty} \int_{supp\, b} \int_{\mathbb{R}^n} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n}}\, dydz$$

$$\lesssim \frac{1}{|x|^n} \|b\|_{L^\infty} \|f\|_{L^{p_1}(w_1)} \sigma_1(B_A(0))^{1/p_1'} \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x|+|x-z|)^n}\, dz,$$

where $\sigma_1 = w_1^{1-p_1'}$ and $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

Sketch of proof for $[T,b]_1$

Proof of condition 2. For the global part of the highlighted term

$$\int_{|z|\geq 1} \frac{|g(z)|}{(|x|+|x-z|)^n}\,dz \leq \|g\|_{L^{p_2}(w_2)} \left(\int_{|z|\geq 1} \frac{\sigma_2(z)}{|z|^{np_2'}}\,dz\right)^{1/p_2'}.$$

Since $w_2 \in A_p \subset A_{p_2}$, we have $\sigma_2 = w_2^{1-p_2'} \in A_{p_2'}$, and then

$$\int_{|z|\geq 1} \frac{\sigma_2(z)}{|z|^{np_2'}}\,dz < \infty.$$

Sketch of proof for $[T,b]_1$

Proof of condition 2. And finally, combining the local and global estimate, rasing both sides to the power p and integrating over $|x| \geq R$, we get

$$\int_{|x|>R} |[T^\delta, b]_1(f,g)(x)|^p \nu_{\vec{w}}\, dx \lesssim_{b,\vec{P},\vec{w}} \int_{|x|>R} \frac{\nu_{\vec{w}}(x)}{|x|^{np}}\, dx \to 0, \quad R \to \infty,$$

where we used again the fact that for $v \in A_r$, $r > 1$,

$$\int_{|x|>R} \frac{v(x)}{|x|^{nr}}\, dx \to 0, \quad R \to \infty.$$

Proof of condition 3.

▸ Smooth truncations

Further remarks

The sufficient condition about $L^p(w)$ precompactness in [CC] may be extended to include weights $w \in A_q$, with $q > p$.

Remark

Let $1 < p < \infty$, $w \in A_\infty$, and $\mathscr{K} \subset L^p(w)$. If

1. $\mathscr{K}$ is bounded in $L^p(w)$;
2. $\lim_{A\to\infty} \int_{|x|>A} |f(x)|^p w dx = 0$ uniformly for $f \in \mathscr{K}$;
3. $\|f(\cdot + t_1) - f(\cdot + t_2)\|_{L^p(w)} \to 0$ uniformly for $f \in \mathscr{K}$ as $|t_1 - t_2| \to 0$;

then $\mathscr{K}$ is precompact.

Further remarks

Observe that

$$\vec{w} \in A_p \times A_p \Rightarrow \vec{w} \in A_{\vec{P}} \text{ and } \nu_{\vec{w}} \in A_p.$$

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Theorem I'

Suppose $\vec{P} \in (1,\infty) \times (1,\infty)$, $p = \frac{p_1 p_2}{p_1 + p_2} > 1$, $b \in CMO$, and $\vec{w} \in A_{\vec{P}}$ with $\nu_{\vec{w}} \in A_p$. Then $[T,b]_1$ and $[T,b]_2$ are compact operators from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.

Theorem II'

Suppose $\vec{P} \in (1,\infty) \times (1,\infty)$, $p = \frac{p_1 p_2}{p_1 + p_2} > 1$, $\vec{b} \in CMO \times CMO$, and $\vec{w} \in A_{\vec{P}}$ with $\nu_{\vec{w}} \in A_p$. Then $[T,\vec{b}]$ is a compact operator from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.

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Smooth truncations

We can also consider smooth truncations. Let $\varphi = \varphi(x,y,z)$ be a non-negative function in $C_c^\infty(\mathbb{R}^{3n})$,

$$supp\, \varphi \subset \{(x,y,z) : \max(|x|,|y|,|z|) < 1\}$$

and such that

$$\int_{\mathbb{R}^{3n}} \varphi(u)\, du = 1.$$

For $\delta > 0$ let $\chi^\delta = \chi^\delta(x,y,z)$ be the characteristic function of the set

$$\{(x,y,z) : \max(|x-y|,|x-z|) \geq \frac{3\delta}{2}\},$$

and let

$$\psi^\delta = \varphi_\delta * \chi^\delta,$$

where

$$\varphi_\delta(x,y,z) = (\delta/4)^{-3n} \varphi(4x/\delta, 4y/\delta, 4z/\delta).$$

Smooth truncations

Clearly we have that $\psi^\delta \in C^\infty$,

$$supp\,\psi^\delta \subset \{(x,y,z) : \max(|x-y|,|x-z|) \geq \delta\},$$

$\psi^\delta(x,y,z) = 1$ if $\max(|x-y|,|x-z|) > 2\delta$, and $\|\psi^\delta\|_{L^\infty} \leq 1$. Moreover, $\nabla\psi^\delta$ is not zero only if $\max(|x-y|,|x-z|) \approx \delta$ and $\|\nabla\psi^\delta\|_{L^\infty} \lesssim 1/\delta$. We define the truncated kernel

$$K^\delta(x,y,z) = \psi^\delta(x,y,z)K(x,y,z).$$

It follows that K^δ satisfies the same size and regularity estimates of K with a constant C independent of δ. As before, we let $T^\delta(f,g)$ be the operator defined pointwise by K^δ, now for all $x \in \mathbb{R}^n$.

▸ Further remarks