Interpolation of the couple $(L \log L, L_{exp})$ and other examples.

VI International Course of Mathematical Analysis in Andalucía

Pedro Fernández Martínez

Universidad de Murcia

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What I will present today is part of following three papers with T.Signes

- Fernández-Martínez, P. and Signes, T., Real interpolation with symmetric spaces and slowly varying functions, *Quart. J. Math.*, **63** No. 1, (2012), 133-164.
- Fernández-Martínez, P. and Signes, T., *Limit cases of reiteration theorems*, to appear in Math. Nachr.
- Fernández-Martínez, P. and Signes, T., Reiteration theorems with extreme values of parameters, *Arkiv der Mathematik*, To appear.

Outline

Some concepts on Interpolation Theory

Slowly varying functions and symmetric spaces

Reiteration Results

Interpolation of $L \log L$ and L_{exp}.

An elementary example

$$Tf(x) = \int_{\mathbb{R}} k(x-y)f(y)dy$$

where $k \in L_q(\mathbb{R})$ for some $1 \leq q \leq \infty$.

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$$T : L_1 \longrightarrow L_q$$

$$\begin{aligned} \|Tf(x)\|_{L_q} &= \| \int_{\mathbb{R}} k(x-y)f(y)dy\|_{L_q} \\ &\leq \int_{\mathbb{R}} \|k(x-y)\|_{L_q} |f(y)|dy \\ &= \|k\|_{L_q}\|f\|_{L_1} \end{aligned}$$

An elementary example

$$Tf(x) = \int_{\mathbb{R}} k(x-y)f(y)dy$$

$$T : L_1 \longrightarrow L_q$$
$$T : L_{q'} \longrightarrow L_\infty$$

$$\|Tf(x)\|_{L_\infty} \leq \|k\|_{L_q} \|f\|_{L_{q'}}$$

An elementary example

$$Tf(x) = \int_{\mathbb{R}} k(x-y)f(y)dy$$

$$\begin{aligned} T :& L_1 \longrightarrow L_q \\ T :& L_{q'} \longrightarrow L_\infty \end{aligned}$$

What can we say about the behavior of T on intermediate spaces?

The Riesz-Thorin interpolation theorem

Theorem

Let $0 < p_0 \neq p_1 < \infty$ *and* $0 < q_0 \neq q_1 < \infty$*, and let*

$$T : L_{p_0} \longrightarrow L_{q_0}$$
$$T : L_{p_1} \longrightarrow L_{q_1}$$

then

$$T : L_p \longrightarrow L_q$$

for $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *and some* $0 < \theta < 1$.

$$\|T\|_{L_p,L_q} \leq \|T\|_{L_{p_0},L_{q_0}}^{1-\theta} \|T\|_{L_{p_1},L_{q_1}}^{\theta}$$

The Real Interpolation Method

If $A_0, A_1 \hookrightarrow \mathcal{U}$ we say that

$$\overline{A} = (A_0, A_1) \quad \text{is an interpolation couple.}$$

$$A_0 \cap A_1 \hookrightarrow A_0 + A_1$$

The Real Interpolation Method

$$A_0 \cap A_1 \hookrightarrow A_0 + A_1$$

$$K(t, a; \overline{A}) = \inf_{a=a_0+a_1} \left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \right\}$$

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Definition

$0 < \theta < 1$ *and* $0 < q \leq \infty$. *The space* $\overline{A}^K_{\theta,q}$ *consists of all those elements s.t.*

$$\|a\|^K_{\theta,q} = \|t^{-\theta} K(t, a)\|_{L_q} < \infty$$

The Real Interpolation Method

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$$\begin{array}{l} T : A_0 \longrightarrow B_0 \\ T : A_1 \longrightarrow B_1 \end{array} \Rightarrow T : \overline{A}^K_{\theta,q} \longrightarrow \overline{B}^K_{\theta,q}$$

The interpolation property

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,q} \hookrightarrow A_0 + A_1$$

$$\begin{aligned} T :& A_0 \longrightarrow B_0 \\ T :& A_1 \longrightarrow B_1 \end{aligned} \Rightarrow T : \overline{A}^K_{\theta,q} \longrightarrow \overline{B}^K_{\theta,q}$$

$$\|T\|_{\overline{A}^K_{\theta,q},\overline{B}^K_{\theta,q}} \leq C\|T\|^{1-\theta}_{A_0,B_0}\|T\|^{\theta}_{A_1,B_1}$$

Example

Consider the couple (L_1, L_∞).

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)ds, \qquad t > 0$$

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$$(L_1, L_\infty)_{1-\frac{1}{p},p} = L_p \qquad 1 < p < \infty$$

The classical reiteration theorem

Let (A_0, A_1) Banach couple, $0 < \theta_0 \neq \theta_1 < 1$ and $0 < q_0, q_1 \leq \infty$.

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Let (A_0, A_1) Banach couple, $0 < \theta_0 \neq \theta_1 < 1$ and $0 < q_0, q_1 \leq \infty$.

$$(\overline{A}^K_{\theta_0,q_0}, \overline{A}^K_{\theta_1,q_1})_{\eta,q}$$

The classical reiteration theorem

Let (A_0, A_1) Banach couple, $0 < \theta_0 \neq \theta_1 < 1$ and $0 < q_0, q_1 \leq \infty$.

Theorem

$$(\overline{A}^K_{\theta_0,q_0}, \overline{A}^K_{\theta_1,q_1})^K_{\eta,q} = \overline{A}^K_{\theta,q}$$

for $0 < \eta < 1$ *and* $\theta = (1-\eta)\theta_0 + \eta\theta_1$

Example

Let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

$$\begin{aligned}(L_{p_0}, L_{p_1})_{\theta,p} &= \left((L_1, L_\infty)_{1-\frac{1}{p_0},p_0}, (L_1, L_\infty)_{1-\frac{1}{p_1},p_1}\right)_{\theta,p} \\ &= (L_1, L_\infty)_{1-(\frac{1-\theta}{p_0}+\frac{\theta}{p_1}),p} \\ &= L_p\end{aligned}$$

Problems

The Lorentz-Zygmund space $L \log L$ is an interpolation space for the couple (L_1, L_∞). However, since $0 < \theta < 1$

$$L \log L \neq (L_1, L_\infty)_{\theta,p}.$$

Similarly

$$L_{exp} \neq (L_1, L_\infty)_{\theta,p}.$$

References

The problem of identifying limit spaces of the real interpolation scale has been studied by several authors:

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Slowly varying functions and symmetric spaces

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,q} \hookrightarrow A_0 + A_1$$

$$\|a\|^K_{\theta,q} = \|t^{-\theta} K(t,a)\|_{L_q} < \infty$$

Slowly varying functions and symmetric spaces

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,\mathbf{b},q} \hookrightarrow A_0 + A_1$$

$$\|a\|^K_{\theta,q} = \|t^{-\theta}\mathbf{b}(\mathbf{t})K(t,a)\|_{L_q} < \infty$$

Slowly varying functions and symmetric spaces

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,\mathbf{b},\mathbf{E}} \hookrightarrow A_0 + A_1$$

$$\|a\|^K_{\theta,q} = \|t^{-\theta}\mathbf{b}(\mathbf{t})K(t,a)\|_{\widetilde{\mathbf{E}}} < \infty$$

for $0 \leq \theta \leq 1$.

Example

Let (Ω, μ) *be a* σ*-finite measure space with* $\mu(\Omega) = 1$*. Then*

$$(L_1, L_\infty)_{0,1,L_1} = L \log L \qquad (L_1, L_\infty)_{1,\ell(t)^{-1},L_1} = L_{exp}.$$

Reiteration Results

Let $\overline{X} = (X_0, X_1)$ be a compatible Banach couple. E_0, E_1 r.i spaces and b_0, b_1 and b slowly varying functions

Reiteration Results

Theorem

$$0 < \theta < 1 \quad \textit{and} \quad \begin{cases} 0 < \theta_0 < \theta_1 < 1 \\ \theta_0 = 0,\ 0 < \theta_1 < 1 \\ 0 < \theta_0 < 1;\ \theta_1 = 1 \end{cases}$$

then

$$\Big((X_0, X_1)_{\theta_0, b_0, E_0}, (X_0, X_1)_{\theta_1, b_1, E_1} \Big)_{\theta, b, E} = (X_0, X_1)_{\tilde{\theta}, \tilde{b}, E}$$

Reiteration Results

Theorem

$$0 \leq \theta < 1 \quad \textit{and} \quad 0 < \theta_1 < 1$$

$$\Big(X_0, (X_0, X_1)_{\theta_1, b_1, E_1}\Big)_{\theta, b, E} = (X_0, X_1)_{\tilde{\theta}, \tilde{b}, E}$$

Reiteration Results

Theorem

$$0 \le \theta < 1 \quad \textit{and} \quad 0 < \theta_1 < 1$$

$$\Big(X_0, (X_0, X_1)_{\theta_1, b_1, E_1}\Big)_{\theta, b, E} = (X_0, X_1)_{\tilde{\theta}, \tilde{b}, E}$$

Theorem

$$0 < \theta \le 1 \quad \textit{and} \quad 0 < \theta_0 < 1$$

$$\Big((X_0, X_1)_{\theta_0, b_0, E_0}, X_1\Big)_{\theta, b, E} = (X_0, X_1)_{\tilde{\theta}, \tilde{b}, E}$$

Reiteration Results

WHAT ABOUT THE CASES

$$\Big((X_0, X_1)_{\theta_0,\mathbf{b}_0,E_0}, (X_0, X_1)_{\theta_1,\mathbf{b}_1,E_1}\Big)_{0,\mathbf{b},E}$$

$$\Big((X_0, X_1)_{\theta_0,\mathbf{b}_0,E_0}, (X_0, X_1)_{\theta_1,\mathbf{b}_1,E_1}\Big)_{1,\mathbf{b},E}.$$

???

Reiteration Results

- $0 \leq \theta \leq 1$
- *a* and **b** slowly varying functions
- *E* and *F* r.i. spaces

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- $0 \leq \theta \leq 1$
- a and $\mathbf{b}$ slowly varying functions
- E and F r.i. spaces

$$(X_0, X_1)^{\mathcal{L}}_{\theta,\mathbf{b},E,a,F} \qquad (X_0, X_1)^{\mathcal{R}}_{\theta,\mathbf{b},E,a,F}$$

Reiteration Results

- $0 \leq \theta \leq 1$
- *a* and **b** slowly varying functions
- *E* and *F* r.i. spaces

$$(X_0, X_1)^{\mathcal{L}}_{\theta,\mathbf{b},E,a,F} \qquad (X_0, X_1)^{\mathcal{R}}_{\theta,\mathbf{b},E,a,F}$$

$$\|f\|^{\mathcal{L}}_{\theta,\mathbf{b},E,a,F} = \left\| \mathbf{b}(t) \| s^{-\theta} a(s) K(s,f) \|_{\widetilde{F}(0,t)} \right\|_{\widetilde{E}}$$

$$\|f\|^{\mathcal{R}}_{\theta,\mathbf{b},E,a,F} = \left\| \mathbf{b}(t) \| s^{-\theta} a(s) K(s,f) \|_{\widetilde{F}(t,\infty)} \right\|_{\widetilde{E}}$$

Reiteration Results

Theorem

$0 < \theta_0 < \theta_1 < 1$.

$$\Big((X_0, X_1)_{\theta_0,\mathbf{b}_0,E_0}, (X_0, X_1)_{\theta_1,\mathbf{b}_1,E_1}\Big)_{0,\mathbf{b},E} = (X_0, X_1)^{\mathcal{L}}{}_{\theta_0,\mathbf{b}\circ\rho,E,\mathbf{b}_0,E_0}.$$

Reiteration Results

Theorem

$0 < \theta_0 < \theta_1 < 1$.

$$\Big((X_0, X_1)_{\theta_0,\mathbf{b}_0,E_0}, (X_0, X_1)_{\theta_1,\mathbf{b}_1,E_1}\Big)_{0,\mathbf{b},E} = (X_0, X_1)^{\mathcal{L}}{}_{\theta_0,\mathbf{b}\circ\rho,E,\mathbf{b}_0,E_0}.$$

$$\Big((X_0, X_1)_{\theta_0,\mathbf{b}_0,E_0}, (X_0, X_1)_{\theta_1,\mathbf{b}_1,E_1}\Big)_{1,\mathbf{b},E} = (X_0, X_1)^{\mathcal{R}}{}_{\theta_1,\mathbf{b}\circ\rho,E,\mathbf{b}_1,E_1}.$$

Reiteration Results

Theorem

$\theta_1 = 1$.

$$\begin{cases} \left(\overline{X}_{\theta_0,\mathbf{b}_0,E_0}, \overline{X}_{1,\mathbf{b}_1,E_1}\right)_{\theta,\mathbf{b},E} = \overline{X}_{\overline{\theta},\overline{\mathbf{b}},E} & \textit{for } 0 < \theta, \theta_1 < 1 \\ \left(\overline{X}_{\theta_0,\mathbf{b}_0,E_0}, \overline{X}_{1,\mathbf{b}_1,E_1}\right)_{0,\mathbf{b},E} = \overline{X}^{\mathcal{L}}_{\theta_0,\mathbf{b}\circ\rho,E,\mathbf{b}_0,E_0} & \\ \left(\overline{X}_{\theta_0,\mathbf{b}_0,E_0}, \overline{X}_{1,\mathbf{b}_1,E_1}\right)_{1,\mathbf{b},E} = \overline{X}_{1,\mathbf{b}^{\#},E} \cap \overline{X}^{\mathcal{R}}_{1,\mathbf{b}\circ\rho,E,\mathbf{b}_1,E_1} & \end{cases}$$

Similar results for $\theta_0 = 0$.

Extreme Reiteration Results

$$\left(\overline{X}_{1,\mathbf{b}_0,E_0}, X_1\right)_{\theta,\mathbf{b},E} = \overline{X}_{1,\tilde{\mathbf{b}},\widehat{E}}$$

$\tilde{\mathbf{b}}(t) = \left(\mathbf{b}_0(t)\varphi_{E_0}(\ell(t))\right)^{1-\theta}\mathbf{b}\left(\mathbf{b}_0(t)\varphi_{E_0}(\ell(t))\right),\ t > 0.$

$$\left(\overline{X}_{1,\mathbf{b}_0,E_0}, X_1\right)_{0,\mathbf{b},E} = \overline{X}^{\mathcal{L}}_{1,\mathbf{b}\circ\rho,\widehat{E},\mathbf{b}_0,E_0}$$

$\rho(t) = \mathbf{b}_0(t)\varphi_{E_0}(\ell(t)),\ t > 0.$

Similar results for $\left(X_0, \overline{X}_{0,\mathbf{b}_1,E_1}\right)$.

Examples

$1 < p < \infty$

$$L_\infty \hookrightarrow L_{exp} \hookrightarrow L_p \hookrightarrow L \log L \hookrightarrow L_1$$

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$$\begin{cases} (L \log L, L_\infty)_{\theta,b,E} & 0 < \theta \leq 1 \\ (L \log L, L_\infty)_{0,b,E} & \end{cases}$$

Examples

$$L_\infty \hookrightarrow L_{exp} \hookrightarrow L_p \hookrightarrow \underset{(L_1,L_\infty)_{0,1,L_1}}{\overset{\displaystyle L\log L}{\|}} \hookrightarrow L_1$$

$$\begin{cases} ((L_1, L_\infty)_{0,1,L_1}, L_\infty)_{\theta,b,E} = L_{\frac{1}{1-\theta},B_\theta,E} & 0 < \theta \leq 1 \\ ((L_1, L_\infty)_{0,1,L_1}, L_\infty)_{0,b,E} = L_{(1,B_0,E)} \cap (L_1, L_\infty)^{\mathcal{L}}_{0,b(t\log 1/t),E,1,L_1} \end{cases}$$

Examples

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Examples

$$L_\infty \hookrightarrow \underset{(L_1,L_\infty)_{1,\,\ell(t)^{-1},L_1}}{\underset{\|}{L_{exp}}} \hookrightarrow L_p \hookrightarrow L\log L \hookrightarrow L_1$$

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Examples

$$L_\infty \hookrightarrow \underset{(L_1,L_\infty)_{1,\ell(t)^{-1},L_1}}{\underset{\parallel}{L_{exp}}} \hookrightarrow L_p \hookrightarrow L\log L \hookrightarrow L_1$$

$$\begin{cases} (L_1, (L_1, L_\infty)_{1,\ell(t)^{-1},L_1})_{\theta,b,E} = L_{q,B_\theta,E} & 0 < \theta < 1 \\ (L_1, (L_1, L_\infty)_{1,\ell(t)^{-1},L_1})_{0,b,E} = L_{(1,B_\theta,E)} \\ (L_1, (L_1, L_\infty)_{1,\ell(t)^{-1},L_1})_{1,b,E} = L_{\infty,\mathbf{B}_1,E} \cap (L_1, L_\infty)^{\mathcal{R}}_{1,\mathbf{b}(t\ell(t)),E,\ell(t)^{-1},L_\infty} \end{cases}$$

Examples

$$L_\infty \hookrightarrow \underset{(L_1, L_\infty)_{1,\,\ell(t)^{-1}, L_1}}{\underset{\|}{L_{exp}}} \hookrightarrow L_p \hookrightarrow L \log L \hookrightarrow L_1$$

$$\begin{cases} (L_1, L_{exp})_{\theta,b,E} = L_{q,B_\theta,E} \qquad 0 < \theta < 1 \\ (L_1, L_{exp})_{0,b,E} = L_{(1,B_\theta,E)} \\ (L_1, L_{exp})_{1,b,E} = L_{\infty,\mathbf{B}_1,E} \cap (L_1, L_\infty)^{\mathcal{R}}_{1,\mathbf{b}(t\ell(t)),E,\ell(t)^{-1},L_\infty} \end{cases}$$

Examples

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$$\begin{cases} \big(L\log L, L_{\exp}\big)_{\theta,\mathbf{b},E} = L_{\frac{1}{1-\theta},\mathbf{B}_\theta,E} & 0<\theta<1 \\ \big(L\log L, L_{\exp}\big)_{0,\mathbf{b},E} = L_{(1,\mathbf{B}_0,E)} \cap (L_1, L_\infty)^{\mathcal{L}}_{0,\mathbf{b}(t\log\frac{1}{t}\ell(t)),E,1,L_1} \\ \big(L\log L, L_{\exp}\big)_{1,\mathbf{b},E} = L_{\infty,\mathbf{B}_1,E} \cap (L_1, L_\infty)^{\mathcal{R}}_{1,\mathbf{b}(t\log\frac{1}{t}\ell(t)),E,\ell(t)^{-1},L_\infty} \end{cases}$$

Extreme cases

$$L_\infty \hookrightarrow L_{exp} \hookrightarrow L_p \hookrightarrow L\log L \hookrightarrow L_1$$

$$\begin{cases} \big(L_1, L\log L\big)_{\theta,\mathbf{b},E} = (L_1, L_\infty)_{0,\tilde{\mathbf{b}},\widehat{E}} & 0 \le \theta < 1 \\ \big(L_1, L\log L\big)_{1,\mathbf{b},E} = (L_1, L_\infty)^{\mathcal{R}}_{0,\tilde{\mathbf{b}},\widehat{E},1,L_1} & \end{cases}$$

The intermediate space $\big(L_1, L\log L\big)_{\theta,1,L_q}$, for $0 < \theta < 1$ and $1 \le q \le \infty$, was identified by Bennett in *Ark. Mat.* (1973).

Extreme cases

$$L_\infty \hookrightarrow L_{exp} \hookrightarrow L_p \hookrightarrow L \log L \hookrightarrow L_1$$

$$\begin{cases} (L_{exp}, L_\infty)_{\theta,\mathbf{b},E} = (L_1, L_\infty)_{1,\tilde{\mathbf{b}},\widehat{E}} & 0 < \theta \leq 1 \\ \big(L_{exp}, L_\infty\big)_{0,\mathbf{b},E} = (L_1, L_\infty)^{\mathcal{L}}_{1,\tilde{\mathbf{b}},\widehat{E},\ell(t)^{-1},L_\infty} & \end{cases}$$

THANK YOU
FOR
YOUR ATTENTION.