On the existence of solutions of differential equations using the coincidence theorems

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Existence and uniqueness to several kinds of differential equations using the Coincidence Theory

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Abstract

The purpose of this article is to study the existence of a coincidence point for two mappings defined on a nonempty set and taking values on a Banach space using the fixed point theory for nonexpansive mappings. Moreover, this type of results will be applied to obtain the existence of solutions for some classes of ordinary differential equations.

Keywords: differential equations, fractional derivative, coincidence problem, fixed point, Ulam-Hyers stability.
MSC: 34A10, 34A08, 47H09

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OUR PROBLEMS

Problem 1. (A three-point BVP of second order)

$$\begin{cases} x''(t) = g\big(t, x(t), x'(t), x''(t)\big) & \text{for a.e. } 0 \le t \le 1, \\ x(0) = 0, \qquad x'(1) = \delta x'(\eta), \end{cases}$$

where $g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, $\delta \neq 1$ and $\eta \in (0,1)$.

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The multi-point boundary value problems for differential equations arise from many fields of applied mathematics and physics. This kind of problems for linear second order ordinary differential equations was initiated in 1987 by Il'in and Moiseev, and motivated by the work of Bitsadze and Samarski on non-local linear elliptic boundary problems.

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A general differential equation with homogeneous Dirichlet condition:

$$\begin{cases} A(u''(t)) - \sin\big(u(t)\big) = g(t), & \text{for } t \in [0,1] \\ u(0) = 0, \quad u(1) = 0, & \end{cases}$$

where the fixed function $g \in \mathcal{C}[0,1]$ is called the driving force, and $A : \mathbb{R} \to \mathbb{R}$ is a certain known function.

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This type of equations is motivated by the study of the forced oscillations of finite amplitude of a pendulum in the absence of a damping force.

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A Cauchy problem with nonlocal initial data for fractional differential equations of Caputo type:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)) & \text{in } \mathbb{R}_+, \\ x(0) = x_0 + g(x), \end{cases}$$

where $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$, $0 < q < 1$, $x_0 \in \mathbb{R}$, and $g(x)$ is defined by $g(x) = \sum_{i=1}^{N} g_i(x(t_i))$.

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Recall that Caputo the fractional derivative of x is defined by

$$^cD^q x(t) := \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x'(s)\, ds,$$

where Γ denotes to the Gamma function.

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Fractional derivatives provide an excellent tool for description of memory and hereditary properties of various materials and processes. This is one of the main advantage of fractional differential equations in comparison with classical integer-order models. A vast collection of real-world problems is drawn form fractional equations of Caputo type.

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COINCIDENCE PROBLEM VS FIXED POINT PROBLEM

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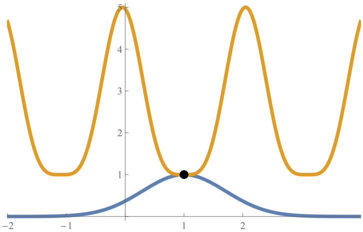

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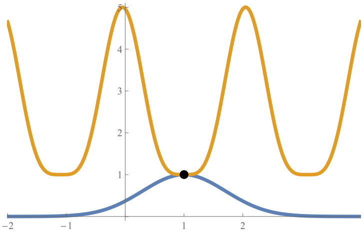

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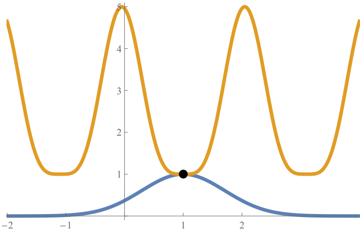

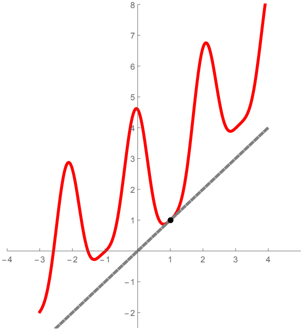

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Recall that, given a normed space $(X, \|\cdot\|)$, a mapping $T : C \subset X \to X$ is said to be **nonexpansive** if

$$\|T(x) - T(y)\| \leq \|x - y\| \qquad \text{for all } x, y \in C.$$

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We say that a Banach space X has the **fixed point property** (**FPP** for short) whenever each nonexpansive sefmapping of each nonempty closed convex bounded subset of X has a fixed point.

COINCIDENCE PROBLEM ASSUMING THE FPP

Theorem 1.

Let X be a nonempty set and $(Y, \|\cdot\|)$ a Banach space with the FPP. Let $T, S : X \to Y$ be two mappings satisfying:

(*i*) $T(X)$ is a closed convex subset of Y,

(*ii*) $S(X) \subset T(X)$ and $\|S(x) - S(y)\| \leq \|T(x) - T(y)\|$ for all $x, y \in X$,

(*iii*) there exist $x_0 \in X$ such that

$$\|T(x) - T(x_0)\| \geq R \Rightarrow S(x) - T(x_0) \neq \lambda(T(x) - T(x_0)) \text{ for all } \lambda > 1.$$

Then there exists at least one x in X such that $Tx = Sx$.

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Question 1.

Give some conditions on X which guarantee that $T(X)$ becomes in closed and convex.

AN APPLICATION TO DIFFERENTIAL EQUATIONS

We can prove that the following three-point boundary value problem has at least one solution $x \in W^{2,2}[0,1]$ such that

$$(P)\begin{cases} x''(t) = g\big(t, x(t), x'(t), x''(t)\big) & \text{for a.e. } 0 \leq t \leq 1, \\ x(0) = 0, \qquad x'(1) = \delta x'(\eta), \end{cases}$$

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Recall the notation of the Sobolev spaces:

$$W^{1,2}[0,1] := \Big\{x : [0,1] \to \mathbb{R} \mid x \text{ abs. cont. on } [0,1] \text{ with } x' \in L^2[0,1]\Big\}$$

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OUR NOTATION

For the sake of simplicity, for any ℓ, we denote by $\mathcal{Z}(\ell)$ the set of non-negative functions $h : [0,1] \to \mathbb{R}_+$ that are Lebesgue integrable on each closed interval contained in $(0,1]$ and satisfy

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By the other hand, if $h : [0,1] \to \mathbb{R}_+$ is a bounded measurable function with its boundedness constant $\kappa > 0$, then $h \in \mathcal{Z}(\frac{\kappa}{4})$.

TWO LEMMAS: THE FIRST ONE

Lemma 1. (Partsvania, 2011)

If $h \in \mathcal{Z}(\ell)$ for some $\ell \geq 0$, then for each $x \in W^{1,2}[0,1]$, with $x(0) = 0$, we have that

$$\int_0^1 h(t)\, x(t)^2 dt \leq 4\ell \int_0^1 x'(t)^2 dt. \tag{1}$$

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If h is a constant function, the inequality (1) is not sharp. Indeed, in this case, we have the well-known Wirtinger inequality. Let $x \in W^{1,2}[0,1]$ be such that $x(0) = 0$. Then

$$\|x\|_2 \leq \frac{2}{\pi} \|x'\|_2 . \qquad (2)$$

TWO LEMMAS: THE SECOND ONE

Lemma 2. (Gupta & Trofimchuk, 1999)

Let $\delta \neq 1$, and $\eta \in (0,1)$ be given. Let $x \in W^{2,2}[0,1]$ be such that $x'(1) = \delta x'(\eta)$. Then

$$\left\|x'\right\|_2 \leq C(\delta,\eta)\,\left\|x''\right\|_2,$$

where

$$C(\delta,\eta) = \begin{cases} \min\left\{\sqrt{F(\delta,\eta)}, \dfrac{2}{\pi}\right\} & \text{if } \delta \leq 0, \\ \sqrt{F(\delta,\eta)} & \text{if } \delta > 0, \end{cases}$$

$$F(\delta,\eta) = \frac{1}{2(\delta-1)^2}\left[\delta^2(1-\eta)^2 + (\delta^2 - 2\delta)\eta^2 + 1\right].$$

A QUESTION IN ORDER TO IMPROVE OUR RESULT

Question 2.

Are there other results similar to Lemma 1 and Lemma 2?

That is, give some conditions such that

- for each $x \in W^{1,2}[0,1]$, with $x(0) = 0$, we have that

$$\int_0^1 h(t)\, x(t)^2 dt \leq K \int_0^1 x'(t)^2 dt.$$

- for each $x \in W^{2,2}[0,1]$, we can ensure that

$$\|x'\|_2 \leq C\, \|x''\|_2\,.$$

Two hypotheses

TWO HYPOTHESES

(H_1) There exist $K_2, K_3 \geq 0$ and $k_1 : [0,1] \to \mathbb{R}$, with $k_1^2 \in \mathcal{Z}(\ell)$ for some $\ell \geq 0$, such that $\left(2\sqrt{\ell} + K_2\right) C(\delta, \eta) + K_3 \leq 1$ and

$$|g(t, u_1, u_2, u_3) - g(t, v_1, v_2, v_3)| \leq k_1(t)\ |u_1 - v_1| + K_2\ |u_2 - v_2| + K_3\ |u_3 - v_3|,$$

for all $t \in [0,1]$ and $u_i, v_i \in \mathbb{R}$ with $i = 1, 2, 3$.

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(H_2) There exist $a_1, a_4 : [0,1] \to \mathbb{R}$ with $a_1^2 \in \mathcal{Z}(m)$ and $a_4 \in L^2[0,1]$, and $A_2, A_3 \geq 0$ such that $\left(2\sqrt{m} + A_2\right) C(\delta, \eta) + A_3 < 1$ and

$$|g(t, u_1, u_2, u_3)| \leq a_1(t)\,|u_1| + A_2\,|u_2| + A_3\,|u_3| + a_4(t)$$

for all $t \in [0,1]$ and $u_i \in \mathbb{R}$ with $i = 1, 2, 3$.

then the problem (P) has at least one solution in $W^{2,2}[0,1]$.

EXAMPLES

Let $\alpha : [0,1] \to \mathbb{R}$ be such that $\alpha^2 \in \mathcal{Z}(\ell)$ for some $\ell \geq 0$. Let $f_2, f_3 : \mathbb{R} \to \mathbb{R}$ be two lipschitzian functions with Lipschitz constant L_2 and L_3, respectively. Let $\beta : [0,1] \to \mathbb{R}$ be a function in $L^2[0,1]$. Consider $g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ defined by

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If $(\frac{3}{2}\sqrt{3\ell} + L_2)C(\delta, \eta) + L_3 \leq 1$ then g satisfies (H_1) and (H_2).

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Example 1.

The problem

$$\begin{cases} \dfrac{x''(t)^3 + 2x''(t)}{x''(t)^2 + 3} = \dfrac{\kappa\, x(t)^2}{t + t\, x(t)^2} + \log\left(t\sqrt{1 + 2e^{x'(t)}}\right) & \text{for } 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = 0, & \end{cases}$$

has at least one solution in $W^{2,2}[0,1]$ whenever $|\kappa| \leq \frac{4\pi - 6}{9\sqrt{3}}$.

COINCIDENCE PROBLEM WITHOUT THE FPP

Recall that (X, d) is a **semi-metric space** if X is a nonempty set and d is a semi-metric, that is, a nonnegative real function $d : X \times X \to \mathbb{R}_+$ such that

(a) $d(x, y) = 0$ if, and only if, $x = y$;

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Note that every metric space (or, more general, every quasi-metric space) is semi-metric but not conversely.

We denote by $\mathcal{F}$ the family of all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(P_1) $f(r) = 0$ if and only if $r = 0$,

(P_2) f is nondecreasing.

Coincidence Problem without the FPP

Theorem 2.

Let (X, d) be a semi-metric space and $(Y, \|\cdot\|)$ a Banach space. Let $T, S : X \to Y$ be two mappings satisfying:

(C_1) $T(X)$ is a closed convex subset of Y,

(C_2) $S(X) \subset T(X)$ and $\|S(x) - S(y)\| \leq \|T(x) - T(y)\|$ for all $x, y \in X$,

(C_3) There exists $f \in \mathcal{F}$ such that $f(\|T(x) - T(y)\|) \leq d(x, y)$ for all $x, y \in X$,

(C_4) $T - S$ is φ-expansive,

(C_5) there exist $x_0 \in X$ such that

$$\|T(x) - T(x_0)\| \geq R \Rightarrow S(x) - T(x_0) \neq \lambda(T(x) - T(x_0)) \text{ for all } \lambda > 1.$$

Then there exists a unique x in X such that $Tx = Sx$.

A generalization of Goebel's Theorem

A GENERALIZATION OF GOEBEL'S THEOREM

Geraghty (1973) gave an interesting generalization of the contraction principle using the class $\boldsymbol{\mathcal{S}}$ of the functions $\alpha : [0, \infty) \to [0, 1)$ such that

$$\lim_{n\to\infty} \alpha(t_n) = 1 \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0. \tag{3}$$

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Using the above result, we can prove a generalization of Goebel's Theorem in the setting of Banach spaces.

Theorem 3.

Let X be a nonempty set, $(Y, \|\cdot\|)$ be a Banach space and $T, S : X \to Y$. Assume that T is onto and there exists a decreasing function $\alpha \in \boldsymbol{\mathcal{S}}$ such that

$$\|Sx - Sy\| \le \alpha\big(\,\|Tx - Ty\|\,\big)\ \|Tx - Ty\| \quad \text{for all } x, y \in X. \tag{4}$$

Then, there exists at least one $x^* \in X$ such that $Tx^* = Sx^*$. If, in addition, T is injective, then the coincidence point x^* is unique.

An application to differential equations

AN APPLICATION TO DIFFERENTIAL EQUATIONS

Consider the Banach space $X := \{u \in \mathcal{C}^2[0,1] : u(0) = u(1) = 0\}$ with the norm $\|u\|_* := \max\{ \|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty \}$

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We can apply Theorem 2 in order to states the existence of classical solutions (on X) for the following general differential equation with homogeneous Dirichlet condition:

$$(P) \begin{cases} A(u''(t)) - \sin\big(u(t)\big) = g(t), & t \in [0,1] \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

where the fixed function $g \in Y$ is called the driving force, and $A : \mathbb{R} \to \mathbb{R}$ is a known function satisfying the following two properties:

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where the fixed function $g \in Y$ is called the driving force, and $A : \mathbb{R} \to \mathbb{R}$ is a known function satisfying the following two properties:

(A_1) A is continuous;

(A_2) there exists a function $f \in \mathcal{F}$ such that

$$f(|Ax - Ay|) \leq |x - y| \leq |Ax - Ay|, \text{ for all } x, y \in \mathbb{R}.$$

Remarks

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$$Ax := \begin{cases} 2\sqrt{x} & \text{if } 0 \leq x \leq 1, \\ kx & \text{if } x > 1, \end{cases}$$

satisfies the property (A_2) with $f(t) = \min\left\{\frac{t^2}{4}, \frac{t}{k}\right\}$.

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Remark

Note that the property (A_1) is necessary, because (A_2) does not imply the continuity of A. Indeed, just take $k > 2$ in the above example.

An application to Fractional Diff. Equations

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)) & \text{in } \mathbb{R}_+, \\ x(0) = x_0 + g(x), \end{cases} \tag{CP}$$

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where $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$, $0 < q < 1$, ${}^cD^q x$ is the Caputo fractional derivative of x, $x_0 \in \mathbb{R}$, and $g(x)$ is defined by

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Notice that g is L_g-lipschitzian with $L_g = \sum_{i=1}^{N} c_i$.

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Recently, N'Guérékata (2009) proved the existence and uniqueness of solutions to problem (CP) on a bounded interval.

Since f is assumed continuous, (CP) is equivalent to the following Volterra integral equation:

$$x(t) = x_0 + g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, x(s)\big)\, ds, \qquad \text{for } t \geq 0. \quad \text{(IE)}$$

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Theorem 3.

Let $0 < q < 1$. Assume that $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function. If there exists a positive constant L_f such that

$$|f(s,u) - f(s,v)| \leq L_f\, |u - v| \qquad \text{for all } u, v \in \mathbb{R} \text{ and a.e. } s \geq 0,$$

then equation (IE) (and, therefore (CP)) has a unique solution in $\mathcal{C}(\mathbb{R}_+)$ whenever

$$\frac{L_f}{\Gamma(q)} \left(\frac{t_N^q}{q} \right) + L_g < 1.$$

REFERENCES

Every result on this talk can be found in the following paper and the references given there.

D. ARIZA-RUIZ, J. GARCÍA-FALSET.
Existence and uniqueness to several kinds of differential equations using the Coincidence Theory.
(submitted to *Taiwanese Journal of Mathematics*)

Thank you

for your attention!