Ultrasymmetric sequence spaces

Joint work with P. Fernández-Martínez

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Summary

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2. Limiting ultrasymmetric sequence spaces
   - The class of functions $\mathcal{P}$
   - Interpolation spaces
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Symmetric spaces

Definition

A Banach sequence space G on $\mathbb{N}$ with the counting measure is symmetric space if it satisfies the condition:

$$\left.\begin{array}{l} a \in G \\ b^*(t) \leq a^*(t) \end{array}\right\} \Longrightarrow b \in G \text{ and } \|b\|_G \leq \|a\|_G$$

where a^* is the decreasing rearrangement of a.

Definition

Let G be a symmetric sequence space. We define the fundamental function of G as

$$\varphi_G(n) = \|e_1 + \ldots + e_n\|_G$$

Remark: We are going to assume that

G has the Fatou property and φ_G is concave

Symmetric spaces

It is known that any symmetric space G is intermediate between the corresponding Lorentz Λ_{φ_G} and Marcinkiewicz M_{φ_G} spaces, defined by

$$\|a\|_{\Lambda_{\varphi_G}} = \sum_{n=1}^{\infty} \frac{\varphi_G(n) a_n^*}{n}, \qquad \|a\|_{M_{\varphi_G}} = \sup_n \varphi_G(n) a_n^{**}$$

where $a_n^{**} = \frac{1}{n} \sum_{k=1}^{n} a_k^*$.

That is,

$$\Lambda_{\varphi_G} \hookrightarrow G \hookrightarrow M_{\varphi_G}$$

and each of the embedding has norm 1.

Example

If $G = \ell_p$*, then* $\varphi_G(n) = n^{1/p}$ *and*

$$\Lambda_{\varphi_G} = \ell^{p,1} \hookrightarrow \ell_p \hookrightarrow M_{\varphi_G} = \ell^{p,\infty}.$$

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$$T : \Lambda_{\varphi_G} \to \Lambda_{\varphi_G} \quad \text{and} \quad T : M_{\varphi_G} \to M_{\varphi_G} \quad \Longrightarrow \quad T : G \to G$$

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Lorentz-Zygmund sequence spaces Bennett and Rudnick (1980)

$$\ell_{p,q}(\log L)^\alpha = (\Lambda_\varphi, M_\varphi)_{1-\frac{1}{q},q}$$

where $\varphi(n) = n^{\frac{1}{p}}(1 + \log n)^\alpha$ and $\ell_{p,q}(\log L)^\alpha$ is the space of sequences a with finite

$$\|a\|_{p,q,\alpha} = \left(\sum_{n=1}^{\infty} \left[n^{\frac{1}{p}} (1 + \log n)^\alpha a_n^* \right]^q \frac{1}{n} \right)^{1/q} \quad \text{if } q < \infty$$

$$\|a\|_{p,\infty,\alpha} = \sup_{n \in \mathbb{N}} n^{\frac{1}{p}} (1 + \log n)^\alpha a_n^{**}.$$

Analytical description

Theorem (Pustylnik, Collect. Math. 2006)

- *If* φ *is a concave function with* $\pi_\varphi > 0$*, and* $\widetilde{E}$ *is a symmetric sequence space with the measure* $\mu(n) = \frac{1}{n}$ *then* $\ell_{\varphi,E}$ *is ultrasymmetric, where*

$$\|a\|_{\ell_{\varphi,E}} := \|\varphi(n)a_n^*\|_{\widetilde{E}}.$$

- *Conversely, any ultrasymmetric sequence space* G *with* $\pi_{\varphi_G} > 0$ *coincides with some* $\ell_{\varphi_G,E}$*.*

Moreover, the space G and $\widetilde{E}$ are related by

$$G = \mathcal{F}(\Lambda_\varphi, M_\varphi) \quad \rightleftharpoons \quad \widetilde{E} = \mathcal{F}(\widetilde{\ell}_1, \ell_\infty)$$

where $\mathcal{F}$ is some interpolation functor and $\|a\|_{\widetilde{\ell}_1} = \sum_{n=1}^\infty |a_n| \frac{1}{n}$.

The Lorentz-Zygmund sequence spaces $\ell_{\infty,q}(\log L)^{\alpha}$ $p < \infty$ are ultrasymmetric.

What happens if $p = \infty$?

Are the Lorentz-Zygmund sequence spaces $\ell_{\infty,q}(\log L)^{\alpha}$ and $\ell^{\alpha}_{\exp}$ ultrasymmetric?

Dilation indices

Let φ be a positive and finite function on $(0,\infty)$. The *dilation function* of φ is

$$m_\varphi(t) = \sup_{\lambda>0} \frac{\varphi(\lambda t)}{\varphi(\lambda)}$$

If m_φ is finite then there exist the lower and upper dilation indices of φ given by

$$\pi_\varphi = \lim_{t\to 0^+} \frac{\log m_\varphi(t)}{\log(t)} \quad \text{and} \quad \rho_\varphi = \lim_{t\to\infty} \frac{\log m_\varphi(t)}{\log(t)}.$$

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It is posible to extend the result to the case

$$0 = \pi_\varphi = \rho_\varphi?$$

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2. *There exist a function* $\Phi : [1, \infty) \longrightarrow (0, \infty)$ *and* $N \in \mathbb{N}$ *such that*

$$\varphi(n) = \Phi(\ell \circ \overset{N}{\cdots} \circ \ell(n))$$

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We denote $L_N(n) = \ell(n) \cdot \Big(\ell \circ \ell(n)\Big) \cdots \Big(\ell \circ \overset{N}{\cdots} \circ \ell(n)\Big)$

The class $\mathcal{P}$

Example 1:

$$\ell^{(\alpha)}(n) = (1 + \log n)^{\alpha}(t) \qquad \text{with} \quad \alpha > 0$$

In this case $\Phi(n) = n^{\alpha}$, and $L_N(n) = \ell(n)$.

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Example 2:

$$\mathcal{L}(n) = (\ell \circ \ell)^{\alpha_2}(n) \cdot (\ell \circ \ell \circ \ell)^{\alpha_3}(n) \qquad \text{with} \quad \alpha_2 > 0$$

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We denote by $\widehat{E}$ the symmetric space E with the measure $\mu(n) = \frac{1}{nL_N(n)}$.

Interpolation spaces

Definition

Let (X_0, X_1) a Banach couple and let $x \in X_0 + X_1$

$$K(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left\{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} \right\}, \quad t > 0.$$

Definition: Let (X_0, X_1) be an ordered Banach couple, $X_0 \hookrightarrow X_1$, and $0 < \theta < 1$. The space $(X_0, X_1)_{\theta,q}$ consists of all those elements in $X_0 + X_1$ such that

$$\|x\|_{(X_0,X_1)_{\theta,q}} = \Big(\sum_{m=0}^{\infty} (2^{-\theta m} K(2^m, x))^q \Big)^{1/q} < \infty$$

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We observe that

$$\left(\sum_{m=0}^{\infty} (2^{-\theta m} K(2^m, x))^q \right)^{1/q} \sim \left(\sum_{n=1}^{\infty} \left(\frac{K(n,x)}{n^\theta} \right)^q \frac{1}{n} \right)^{1/q} = \left\| \frac{K(n,x)}{n^\theta} \right\|_{\tilde{\ell}_q}$$

Definition

Let (X_0, X_1) *be an ordered Banach couple,* $X_0 \hookrightarrow X_1$*, let* $\varphi \in \mathcal{P}$ *and let* $\widehat{E}$ *a symmetric sequence space with the measure* $\mu(n) = \frac{1}{nL_N(n)}$*. The space* $(X_0, X_1)_{\frac{n}{\varphi(n)}, \widehat{E}}$ *consists of all those elements in* $X_0 + X_1$ *such that*

$$\|x\|_{(X_0,X_1)_{\frac{n}{\varphi(n)}, \widehat{E}}} = \left\| \left\{ \frac{K(n,x)}{n/\varphi(n)} \right\}_{n\in\mathbb{N}} \right\|_{\widehat{E}} < \infty$$

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Theorem

If $\varphi \in \mathcal{P}$ *then*

$$\Lambda_\varphi = (\ell_1, \ell_\infty)_{\frac{n}{\varphi(n)}, \widehat{\ell_1}} \qquad M_\varphi = (\ell_1, \ell_\infty)_{\frac{n}{\varphi(n)}, \ell_\infty}.$$

Reiteration theorem

Theorem (Reiteration Theorem)

Let $\mathcal{F}$ be an interpolation functor, and let (X_0, X_1) be an ordered Banach couple. If $\varphi \in \mathcal{P}$, then

$$\mathcal{F}\left((X_0, X_1)_{\frac{n}{\varphi(n)},\widehat{\ell}_1}, (X_0, X_1)_{\frac{n}{\varphi(n)},\ell_\infty}\right) = (X_0, X_1)_{\frac{n}{\varphi(n)},\mathcal{F}(\widehat{\ell}_1,\ell_\infty)}$$

Analytical description in the limit case

Theorem

A symmetric sequence space G, with fundamental function $\varphi_G \in \mathcal{P}$, is ultrasymmetric if and only if

$$\|a\|_G \sim \|\varphi(n) a^*(t)\|_{\widehat{E}}$$

where $\widehat{E}$ is a symmetric sequence space with respect to the counting measure $\mu(n) = \frac{1}{nL_N(n)}$, for any $\varphi \sim \varphi_G$ and

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Moreover, the space G and $\widehat{E}$ are related by

$$G = \mathcal{F}(\Lambda_\varphi, M_\varphi) \quad \rightleftharpoons \quad \widehat{E} = \mathcal{F}\big(\widehat{\ell}_1, \ell_\infty\big),$$

where $\mathcal{F}$ is some interpolation functor and

$$\|a\|_{\widehat{\ell}_1} = \sum_{n=1}^{\infty} |a_n| \frac{1}{nL_N(n)}.$$

Examples

The Lorentz-Zygmund sequence spaces $\ell_{\infty,q}(\log L)^\alpha$ and $\ell_{\exp}^\alpha$ are ultrasymmetric.

Indeed, if $\alpha + \frac{1}{q} < 0$ then

$$\|a\|_{\infty,q,\alpha} = \left(\sum_{n=1}^{\infty} \left[(1+\log n)^\alpha a_n^* \right]^q \frac{1}{n} \right)^{1/q} = \left(\sum_{n=1}^{\infty} \left[(1+\log n)^{\alpha+\frac{1}{q}} a_n^* \right]^q \frac{1}{\ell(n)} \right)^{1/q}$$

so

$$\ell_{\infty,q}(\log L)^\alpha = (\Lambda_\varphi, M_\varphi)_{1-\frac{1}{q},q} \quad \text{where} \quad \varphi(n) = (1+\log(n))^{\alpha+\frac{1}{q}} \in \mathcal{P}.$$

Observe that $\widehat{L}_q = (\widehat{L}_1, L_\infty)_{1-\frac{1}{q},q}$.

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so

$$\ell_{\infty,q}(\log L)^{\alpha} = (\Lambda_{\varphi}, M_{\varphi})_{1-\frac{1}{q},q} \quad \text{where} \quad \varphi(n) = (1 + \log(n))^{\alpha + \frac{1}{q}} \in \mathcal{P}.$$

Observe that $\widehat{L}_q = (\widehat{L}_1, L_{\infty})_{1-\frac{1}{q},q}$.
Similarly, if $\alpha > 0$ then

$$\ell_{\exp}^{\alpha} = \ell_{\infty,\infty}(\log L)^{\alpha} = (\Lambda_{\varphi}, M_{\varphi})_{1,\infty} \quad \text{where} \quad \varphi(n) = (1 + \log(n))^{\alpha} \in \mathcal{P}.$$

Limit ultrasymmetric operator ideals

The approximation numbers $(a_n(T))$ of a bounded linear operator $T \in L(X, Y)$ acting between the Banach spaces X an Y, are given by

$$a_n(T) = \inf\{\|T - T_n\| : T_n \in L(X, Y), rank T_n < n\}, \quad n = 1, 2, \ldots.$$

Definition

Let $\varphi \in \mathcal{P}$ and let $\widehat{E}$ be a symmetric sequence space with the measure $\mu(t) = \frac{1}{nL_n(n)}$. We define the limit ultrasymmetric operator ideal $\mathcal{L}_{\varphi,\widehat{E}}(X, Y)$ as the set of all those elements $T \in L(X, Y)$, which have a finite norm

$$\|T\|_{\mathcal{L}_{\varphi,\widehat{E}}(X,Y)} = \|\varphi(n)a_n(f)\|_{\widehat{E}}.$$

Particula case: Lorentz-Zygmund operator ideals $\mathcal{L}_{\infty,q,\gamma}(X, Y)$ ideals

Representation theorem

Theorem

Let $\varphi \in \mathcal{P}$ and let E be a symmetric sequence space. Then the following are equivalent for $f \in L(X, Y)$.

1. $f \in \mathcal{L}_{\varphi,\widehat{E}}(X, Y)$.
2. *There exists a sequence* $(T_{\lambda_n})_{n=0}^{\infty} \subseteq L(X, Y)$, *such that* $\mathrm{rank} T_n \leq \lambda_n$, $T = \sum_{n=0}^{\infty} T_{\lambda_n}$ *converges in the operator norm and*

$$\Big\| \varphi(\lambda_n) \| T_{\lambda_n} \|_{L(X,Y)} \Big\|_E < \infty.$$

In this case

$$\|f\|_{X_{\varphi,\widehat{E}}} \sim \inf \left\{ \Big\| \varphi(\lambda_n) \| T_{\lambda_n} \|_X \Big\|_E \right\}$$

where λ_n are the iterated potential functions $\lambda_n = A_N(2^n)$ *with*

$$A_1(s) = 2^{s-1}, \quad A_n(s) = A_{n-1}(2^{s-1}).$$

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