Convex inequalities, isoperimetry and spectral gap II

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Part II. Isoperimetric inequalities

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- Log-concave case, Cheeger isoperimetric type inequalities
- Cheeger versus Poincaré inequalites
- E. Milman's theorem

Isoperimetric versus functional inequalities

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The classical isoperimetric inequality in

$$|\partial A| \geq C\,|A|^{1-\frac{1}{n}} \qquad \forall \text{ bounded borel } A \subseteq R^n$$

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$|\partial A|$ is the outer Minkowski content of A,defined by

$$|\partial A| = \liminf_{\varepsilon \to 0} \frac{|A^\varepsilon| - |A|}{\varepsilon},$$

where

$$A^\varepsilon = \{a + x; a \in A, |x| < \varepsilon\} = A + \varepsilon B$$

is the ε-dilation of A. The outer Minkowski content coincides with the $(n-1)$-dimensional Hausdorff measure of the boundary for bounded borelians with smooth enough boundary.

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The constant

$$C = \frac{|S^{n-1}|}{|B|^{1-\frac{1}{n}}}$$

Isoperimetric inequality $\Longleftrightarrow$ Sobolev inequality

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TFAE with the same constant C:

- For any bounded borelian $A \subset \mathbb{R}^n$

$$|\partial A| \geq C\,|A|^{1-\frac{1}{n}}$$

- $$\|\,|\nabla f|\,\|_1 \geq C\|f\|_{\frac{n}{n-1}}$$

  for any locally Lipschitz compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$

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Here

$$\|f\|_{\frac{n}{n-1}} = \left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}.$$

and

$$|\nabla f(x)| = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

Isoperimetric inequalities for log-concave probabilities

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Given a convex body in $K \subset \mathbb{R}^n$ find a surface which divide K into two parts whose measure is minimun relative to the volume of the two parts

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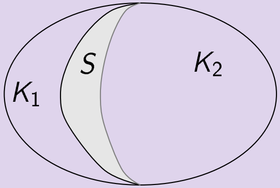

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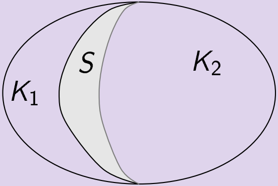

$$\mathrm{vol}_{n-1}(\partial_K S) \geq C \frac{\mathrm{vol}(K_1) \cdot \mathrm{vol}(K_2)}{\mathrm{vol}(K)}$$

If we normalize $\mu(A) = \dfrac{|A|}{|K|}$

$$\mathrm{vol}_{n-1}(\partial_K S) \geq C \, \frac{\mathrm{vol}(K_1)\cdot \mathrm{vol}(K_2)}{\mathrm{vol}(K)}$$

$$\Updownarrow$$

$$\mu^+(A) \geq C \, \mu(A)\mu(A^c)$$

Cheeger's type isoperimetric inequality
(in Riemannian geometry)

Cheeger isoperimetric problem for log-concave probabilities

Given μ (the uniform probability on a convex body K or more generally any log-concave probability), estimate the best constant $C > 0$ for which

$$\mu^+(A) \geq C\mu(A)\mu(A^c) \qquad \forall\, A \text{ borelian } \subseteq \mathbb{R}^n$$

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$$\mu^+(A) \geq C' \min\{\mu(A), \mu(A^c)\} \qquad \forall\, A \text{ borelian } \subseteq \mathbb{R}^n$$

$$C \geq C' \geq \frac{C}{2}$$

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where

$$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon}$$

and

$$A^\varepsilon = \{a + x; a \in A, |x| < \varepsilon\}$$

Cheeger versus Poincaré type inequalities

Theorem. (Maz'ja, Cheeger)

Let μ be a Borel probability measure in $\mathbb{R}^n$. The following statements are equivalent:

i) For any Borel set $A \subseteq \mathbb{R}^n$

$$\mu^+(A) \geq C_1 \min\{\mu(A), \mu(A^c)\}$$

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$$C_2 \|f - \mathbb{E}_\mu f\|_1 \leq \| \, |\nabla f| \, \|_1 .$$

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Moreover $C_2 \leq C_1 \leq 2C_2$.
Notation

$$\mathbb{E}_\mu f = \int f\, d\mu \qquad \text{and} \qquad \|g\|_1 = \mathbb{E}_\mu|g| = \int |g| d\mu$$

Proof: i) $\implies$ ii)

We use the coarea formula

Coarea formula

Let $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $g : \mathbb{R}^n \to [0, \infty)$ measurable, then

$$\int_{\mathbb{R}^n} g(x) \cdot |\nabla f(x)| dx = \int_{-\infty}^{\infty} \int_{f(x)=t} g(f^{-1}(t)) d\mathcal{H}_{n-1}\, dt$$

If $g = 1$ then

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f(x)| dx &= \int_{-\infty}^{\infty} \int_{f(x)=t} d\mathcal{H}_{n-1}\, dt \\ &= \int_{-\infty}^{\infty} \mathcal{H}_{n-1}\{x \in \mathbb{R}^n; f(x) = t\} dt\end{aligned}$$

Coarea formula II

Assume $f > 0$ locally Lipschitz and μ a log-concave probability

$$\int_{\mathbb{R}^n} |\nabla f(x)| d\mu(x) \geq \int_0^\infty \mu^+\{f > t\} dt$$

where $A_t = \{x \in \mathbb{R}^n : f(x) > t\}$

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$$\geq \text{ (by i)}$$
$$\geq C_1 \int_0^\infty \min\{\mu(A_t), \mu(A_t^c)\} dt \geq (*)$$

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where $A_t = \{x \in \mathbb{R}^n : f(x) > t\}$ and so,

$$(*) \geq C_1 \int_0^\infty \mu(A_t)\mu(A_t^c) dt = \frac{C_1}{2} \int_0^\infty \|\chi_{A_t} - \mathbb{E}_\mu \chi_{A_t}\|_1 dt$$
$$= \frac{C_1}{2} \int_0^\infty \sup_{\|g\|_\infty = 1} \int_{\mathbb{R}^n} (\chi_{A_t}(x) - \mathbb{E}_\mu \chi_{A_t}) g(x) d\mu(x) dt$$
$$\geq \frac{C_1}{2} \sup_{\|g\|_\infty = 1} \int_0^\infty \int_{\mathbb{R}^n} (\chi_{A_t}(x) - \mathbb{E}_\mu \chi_{A_t}) g(x) d\mu(x) dt = (**)$$

So

$$
\begin{aligned}
(**) &= \frac{C_1}{2} \sup_{\|g\|_\infty = 1} \int_0^\infty \int_{\mathbb{R}^n} \chi_{A_t}(x)(g(x) - \mathbb{E}_\mu g) d\mu(x) dt \\
&= \frac{C_1}{2} \sup_{\|g\|_\infty = 1} \int_{\mathbb{R}^n} (g(x) - \mathbb{E}_\mu g) f(x) d\mu = \frac{C_1}{2} \|f - \mathbb{E}_\mu f\|_1 .
\end{aligned}
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So

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$$= \frac{C_1}{2} \sup_{\|g\|_\infty = 1} \int_{\mathbb{R}^n} (g(x) - \mathbb{E}_\mu g) f(x) d\mu = \frac{C_1}{2} \|f - \mathbb{E}_\mu f\|_1.$$

In the general case we proceed for bounded below functions and then by an approximation argument.

Proof: ii) $\Longrightarrow$ i)

Let A be a Borel set in $\mathbb{R}^n$. Given $0 < \varepsilon < 1$, we define

$$f_\varepsilon(x) = \max\left\{0, 1 - \frac{d(x, A^{\varepsilon^2})}{\varepsilon - \varepsilon^2}\right\}.$$

Proof: ii) $\Longrightarrow$ i)

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$$f_\varepsilon(x) = \max\left\{0, 1 - \frac{d(x, A^{\varepsilon^2})}{\varepsilon - \varepsilon^2}\right\}.$$

- $0 \leq f_\varepsilon(x) \leq 1$
- $f_\varepsilon(x) = 1$ if $x \in A^{\varepsilon^2} (\supseteq A)$
- $f(x) = 0$, whenever $d(x, A) > \varepsilon$
- $\lim_{\varepsilon \to 0} f_\varepsilon = \chi_{\overline{A}}$
- $$|f_\varepsilon(x) - f_\varepsilon(y)| \leq \frac{1}{\varepsilon(1-\varepsilon)} \left| d(x, A^{\varepsilon^2}) - d(y, A^{\varepsilon^2}) \right| \leq \frac{|x-y|}{\varepsilon(1-\varepsilon)}$$
- f_ε is locally Lipschtiz and

$$|\nabla f_\varepsilon(x)| \leq \frac{1}{\varepsilon - \varepsilon^2} \qquad x \in \mathbb{R}^n.$$

$|\nabla f_\varepsilon(x)| = 0$ whenever
$x \in \{x \in \mathbb{R}^n; d(x, A) > \varepsilon\} \cup A^{\varepsilon^2} \supseteq \{x \in \mathbb{R}^n; d(x, A) > \varepsilon\} \cup A$.
Thus,

$$\int_{\mathbb{R}^n} |\nabla f_\varepsilon(x)| d\mu(x) \leq \frac{\mu(A^{\varepsilon+\varepsilon^2}) - \mu(A)}{\varepsilon - \varepsilon^2}.$$

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Thus,

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By *ii)*,

$$C_2 \|f_\varepsilon - \mathbb{E}_\mu f_\varepsilon\|_1 \leq \frac{\mu(A^{\varepsilon+\varepsilon^2}) - \mu(A)}{\varepsilon - \varepsilon^2}$$

and letting $\varepsilon \to 0^+$ we obtain

$$2C_2 \mu(A)\mu(A^c) \leq \mu^+(A).$$

this gives

$$C_2 \min\{\mu(A), \mu(A^c)\} \leq \mu^+(A).$$

Poincaré inequalities associated to a log-concave μ

Given $1 \leq p \leq q \leq \infty$, we introduce

$D_{p,q}(\mu)$

The biggest constant por the inequality

$$D_{p,q} \|f - \mathbb{E}_\mu f\|_p \leq \|\,|\nabla f|\,\|_q$$

for any locally Lipschitz integrable functions $f \in L^p(d\mu)$.

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- Case $p = q = 1$: $D_{1,1}$ is equivalent to the isoperimetric Cheeger constant for μ

- Case $p = q = 2$: is the Poincaré inequality for $d\mu = e^{-V(x)}dx$.

$$\Updownarrow$$

$$D_{2,2}^2 \underbrace{\int_{\mathbb{R}^n} \left|f - \int_{\mathbb{R}^n} f\, d\mu\right|^2 d\mu}_{\mathrm{Var}_\mu(f)} \leq \int_{\mathbb{R}^n} |\nabla f|^2\, d\mu$$

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- $D_{2,2}^2 = \lambda_2$ is known as spectral gap of μ. $=$ first eigenvalue of the Laplace Beltrami operator

$$L = \Delta - \langle \nabla V, \nabla \rangle$$

Relations

- By Hölder's inequality, if $1 \leq p \leq q \leq \infty$

$$D_{p,q} \leq D_{p,\infty} \leq D_{1,\infty}$$

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- Maz'ja and Cheeger (1960)

$$D_{1,1} \leq CD_{2,2}$$

C absolute constant

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- Easy modification by Hölder

$$D_{p,p} \leq Cp' D_{p',p'} \qquad \forall 1 \leq p \leq p' \leq \infty,$$

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- Ledoux (1994) using the semigroup of operators associated to L

$$D_{2,2} \le CD_{1,1}$$

- E. Milman (2010)

$$D_{1,\infty} \leq C D_{1,1}$$

In consequence

$$D_{p,q} \leq D_{1,\infty} \leq C D_{1,1} \leq C\, p\, D_{p,q}$$

C absolute (independent of μ and even of the dimension)

Equivalence

Theorem (E.Milman, 2010)

Let μ be a log-concave probability in $\mathbb{R}^n$. Assume that

$$D_{1,\infty}\mathbb{E}_\mu|f - \mathbb{E}_\mu f| \leq \| \, |\nabla f| \, \|_\infty \qquad \forall f$$

Then

$$\mu^+(A) \geq C D_{1,\infty}\mu(A)^2 \qquad \forall \mu(A) \leq \frac{1}{2}$$

where $C > 0$ is an absolute constant.

Equivalence

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Then

$$\mu^+(A) \geq C D_{1,\infty}\mu(A)^2 \qquad \forall \mu(A) \leq \frac{1}{2}$$

where $C > 0$ is an absolute constant. Moreover

$$C D_{1,\infty}\mathbb{E}_\mu|f - \mathbb{E}_\mu f| \leq \mathbb{E}_\mu|\nabla f|$$

and

$$C D_{1,\infty} \leq D_{1,1}$$

Moreover part: $\mu^+(A) \geq CD_{1,\infty}\mu(A)^2$ is enough

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$$\mu(A) = \frac{1}{2} \Longrightarrow \mu^+(A) \geq C \frac{D_{1,\infty}}{4}$$

The isoperimetric profile

$$I_\mu(t) := \inf\{\mu^+(A) \, : \, \mu(A) = t\}, \qquad 0 \leq t \leq \frac{1}{2}$$

is a concave function (absolutely non trivial fact!).

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Bavard-Pansu, Bérard-Besson-Gallot, Gallot, Morgan-Johnson, Sternberg-Zumbrun, Kuwert, Bayle-Rosales, Bayle, Morgan , Bobkov

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Given $0 \leq t \leq \frac{1}{2}$

$$I_\mu(t) \geq 2tI_\mu\left(\frac{1}{2}\right) \geq 2tCD_{1,\infty}\frac{1}{4} \geq CD_{1,\infty}\frac{t}{2}$$

then

$$\mu^+(A) \geq CD_{1,\infty}\mu(A) \qquad \text{whenever, } \mu(A) \leq \frac{1}{2}$$

Hence, Cheeger's theorem implies

$$\mathbb{E}_\mu|\nabla f| \geq CD_{1,\infty}\mathbb{E}_\mu|f - \mathbb{E}_f|.$$

Semigroups technique (Ledoux)

Given $d\mu = e^{-V(x)}dx$, V convex and smooth, the associated Laplace-Beltrami operator

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Given $d\mu = e^{-V(x)}dx$, V convex and smooth, the associated Laplace-Beltrami operator

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Let $(P_t)_{t\geq 0}$ be the semigroup generated by L
It is characterized by the following system of differential equations of second order

$$\frac{d}{dt}P_t(f) = L(P_t(f))$$
$$P_0(f) = f$$

for every bounded smooth function f.

Properties

1) $P_t(1) = 1$
2) $f \geq 0 \Longrightarrow P_t(f) \geq 0$
3) $\mathbb{E}_\mu P_t(f) = \mathbb{E}_\mu f$
4) $\mathbb{E}_\mu |P_t(f)|^p \leq \mathbb{E}_\mu |f|^p, \ \forall p \geq 1$

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4) $\mathbb{E}_\mu |P_t(f)|^p \leq \mathbb{E}_\mu |f|^p$, $\forall p \geq 1$
5) (Bakry-Ledoux).
   If $2 \leq q \leq \infty$ and f bounded and smooth

$$\| \, |\nabla P_t(f)| \, \|_{L^q(\mu)} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^q(\mu)}.$$

6) (Ledoux) If f bounded smooth

$$\|f - P_t(f)\|_{L^1(\mu)} \leq \sqrt{2t} \| \, |\nabla f| \, \|_{L^1(\mu)}.$$

Milman's proof

Assume that A is closed, $\mu(A) \leq \frac{1}{2}$. Given $\varepsilon > 0$, let
$A^{\varepsilon} = \{x \in \mathbb{R}^n \ : \ d(x, A) < \varepsilon\}$
$\chi_{A,\varepsilon}(x) = \max\{1 - \frac{1}{\varepsilon} d(x, A), 0\}$ Lipschitz

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- $\lim_{\varepsilon \to 0} \chi_{A,\varepsilon}(x) = 1 \iff x \in A$
-

$$|\nabla \chi_{A,\varepsilon}(x)| = \begin{cases} = 0, & \text{if} \quad x \in \mathrm{int} A \\ = 0, & \text{if} \quad d(x, A) > \varepsilon \\ \leq \frac{1}{\varepsilon}, & \text{if} \quad x \notin \mathrm{int} A, 0 \leq d(x, A) \leq \varepsilon \end{cases}$$

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Then

$$\frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon} \geq \int_{\mathbb{R}^n} |\nabla \chi_{A,\varepsilon}(x)| d\mu(x)$$
$$\geq \text{(by 6)} \geq \frac{1}{\sqrt{2t}} \mathbb{E}_\mu |\chi_{A,\varepsilon} - P_t(\chi_{A,\varepsilon})|$$

When $\varepsilon \to 0$

$$\begin{aligned}
\sqrt{2t}\,\mu^{+}(A) &\geq \mathbb{E}_\mu|\chi_A - P_t(\chi_A)| = 2\left(\mu(A) - \int_A P_t(\chi_A)(x)d\mu(x)\right) \\
&= 2\big(\mu(A)\mu(A^c) - \mathbb{E}_\mu(\chi_A - \mu(A))(P_t(\chi_A) - \mu(A))\big) \\
&\leq \text{(by Hölder)} \\
&\geq 2\big(\mu(A)\mu(A^c) - \|\chi_A - \mu(A)\|_\infty \mathbb{E}_\mu|P_t(\chi_A) - \mu(A)|\big)
\end{aligned}$$

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$$= 2\big(\mu(A)\mu(A^c) - \mathbb{E}_\mu(\chi_A - \mu(A))(P_t(\chi_A) - \mu(A))\big)$$
$$\leq \text{(by Hölder)}$$
$$\geq 2\big(\mu(A)\mu(A^c) - \|\chi_A - \mu(A)\|_\infty \mathbb{E}_\mu|P_t(\chi_A) - \mu(A)|\big)$$

We use the hypothesis

$$\mathbb{E}_\mu|P_t(\chi_A) - \mu(A)| \leq \frac{1}{D_{1,\infty}}|||\nabla P_t(\chi_A)|||_\infty$$

since $\mathbb{E}_\mu P_t(\chi_A) = \mu(A)$.

$\nabla P_t(\chi_A - \mu(A)) = \nabla P_t(\chi_A)$, we have

$$\begin{aligned}
\||\nabla P_t(\chi_A)|\|_\infty &\leq \||\nabla P_t(\chi_A)|\|_\infty \\
&\leq \text{ (by 5)} \\
&\leq \frac{1}{\sqrt{2t}} \|P_t \chi_A - \mu(A)\|_\infty \\
&\leq (\text{ by 4}) \\
&\leq \frac{1}{\sqrt{2t}} \|\chi_A - \mu(A)\|_\infty \leq \frac{1}{\sqrt{2t}}
\end{aligned}$$

$\nabla P_t(\chi_A - \mu(A)) = \nabla P_t(\chi_A)$, we have

$$\begin{aligned} \||\nabla P_t(\chi_A)|\|_\infty &\leq \||\nabla P_t(\chi_A)|\|_\infty \\ &\leq \text{ (by 5)} \\ &\leq \frac{1}{\sqrt{2t}}\|P_t\chi_A - \mu(A)\|_\infty \\ &\leq (\text{ by 4}) \\ &\leq \frac{1}{\sqrt{2t}}\|\chi_A - \mu(A)\|_\infty \leq \frac{1}{\sqrt{2t}} \end{aligned}$$

Since $\mu(A) \leq \frac{1}{2}$,

$$\sqrt{2t}\mu^+(A) \geq \mu(A) - \frac{2}{\sqrt{2t}D_{1,\infty}}$$

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Choose $\sqrt{2t}D_{1,\infty} = 4/\mu(A)$ and we get

$$\mu^+(A) \ge 8D_{1,\infty}\mu(A)^2.$$