On frequently hypercyclic operators

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Introduction

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Two recent textbooks on Linear Dynamics:

F. Bayart, E. Matheron: *The dynamics of linear operators*, CUP 2009

KGE, A. Peris: *Linear chaos*, Springer 2011

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- T is called chaotic if it is hypercyclic and it has a dense set of periodic points.

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Recall that, for $A \subset \mathbb{N}_0$,

$$\underline{\text{dens}}\, A = \liminf_{N \to \infty} \frac{\#\{n \leq N : n \in A\}}{N+1}.$$

Equivalently, x is frequently hypercyclic for T if, for any non-empty open subset U of X,

$$\exists (n_k) \text{ with } n_k = O(k) \text{ such that } T^{n_k} x \in U.$$

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for any non-empty open subset U of X, there is some M such that
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Central question:

What is the relationship between frequent hypercyclicity and chaos?

Topological approach to frequent hypercyclicity

How to detect if an operator is frequently hypercyclic?

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(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(ii) $\sum_{n=0}^{\infty} S^n x$ *converges unconditionally,*

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Then T is frequently hypercyclic and chaotic.

Example

Let $B(x_n) = (x_{n+1})$ be the backward shift on ℓ^p, $1 \leq p < \infty$, or c_0.
Then λB is frequently hypercyclic if (and only if) $|\lambda| > 1$.

More generally, let

$$B_w(x_n) = (w_2x_2, w_3x_3, w_4x_4, \dots)$$

be a weighted backward shift, with $\sup_n |w_n| < \infty$. Then $B_w : \ell^p \to \ell^p$ and $B_w : c_0 \to c_0$.

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Theorem

In the following, each assertion implies the next:

(i)

$$\sum_{n=1}^{\infty} \frac{1}{|w_2 w_3 \dots w_n|^p} < \infty;$$

($\iff$ B_w is chaotic)

(ii) *B_w satisfies the Frequent Hypercyclicity Criterion;*

(iii) *B_w is frequently hypercyclic.*

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Theorem

The following assertions are equivalent:

(i)
$$\sum_{n=1}^{\infty} \frac{1}{|w_2 w_3 \dots w_n|^p} < \infty;$$

($\Longleftrightarrow$ B_w is chaotic)

(ii) *B_w satisfies the Frequent Hypercyclicity Criterion;*

(iii) *B_w is frequently hypercyclic.*

The implication (iii)$\Longrightarrow$(i) is due to Bayart and Ruzsa (2013).

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Example (Bayart-Grivaux 2007)

There is a weighted backward shift B_w on c_0 that is frequently hypercyclic but that does not have any periodic points, and which is therefore not chaotic.

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Which (strong) dynamical behaviour does the Frequent Hypercyclicity Criterion characterize?

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Question 1

Which (strong) dynamical behaviour does the Frequent Hypercyclicity Criterion characterize?
Does every chaotic and frequently hypercyclic operator satisfy the Frequent Hypercyclicity Criterion?

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For any hypercyclic operator T, the set

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Consequence

Every vector in X is the sum of two hypercyclic vectors for T:

$$X = HC(T) + HC(T).$$

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Indeed, for any $x \in X$, $HC(T) \cap (x - HC(T)) \neq \varnothing$.

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Theorem (Moothathu 2013, Bayart-Ruzsa 2013, Grivaux-Matheron 2014)

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$$T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}), \quad Tf(z) = f(z+1),$$

we have that

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Question 2 (Antequera 2006, Bonilla-GE 2007)

Is there a Banach space operator T for which $X = FHC(T) + FHC(T)$?

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We refer to the deep work of Bayart, Grivaux and Matheron:

S. Grivaux: A new class of frequently hypercyclic operators, Indiana Univ. Math. J. 2011

S. Grivaux, E. Matheron: Invariant measures for frequently hypercyclic operators, Adv. Math. 2014

F. Bayart, E. Matheron: Mixing operators and small subsets of the circle, J. reine angew. Math. 2014

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For any frequently hypercyclic operator, the set $FHC(T)$ contains a dense subspace (except 0). In other words, $FHC(T)$ is densely lineable.

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Suppose that an operator T on a separable Banach space X
- *satisfies the Frequent Hypercyclicity Criterion,*
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Then there is a closed infinite-dimensional subspace in which any vector (except 0) is frequently hypercyclic for T; i.e. $FHC(T)$ is spaceable.

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The operator T on $C_0(\mathbb{R}_+)$,

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Theorem (Menet 2014)

YES. In fact, there is a weighted backward shifts B_w on ℓ^p with this property.

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Question 4 (Menet 2014)

Characterize the weighted backward shifts B_w on ℓ^p that possess a frequently hypercyclic subspace.

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Question 4 (Menet 2014)

Characterize the weighted backward shifts B_w on ℓ^p that possess a frequently hypercyclic subspace.

For hypercyclic subspaces, a characterization is due to González, León and Montes (2000) in the complex case, and Menet (2014) in the real case.

Existence of frequently hypercyclic operators

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How about frequent hypercyclicity?

The key is the following.

Theorem (Shkarin 2009)

Let T be a frequently hypercyclic operator on a complex Banach space. Then its spectrum $\sigma(T)$ has no isolated points.

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Question 5

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It follows from Shkarin's result and the Riesz theory that no operator of the form

$$T = \lambda I + K, \quad K \text{ compact}$$

can be frequently hypercyclic.

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Argyros-Haydon spaces do not admit frequently hypercyclic operators.

On the positive side we have the following.

Theorem (de la Rosa-Frerick-Grivaux-Peris 2012)

Every complex separable infinite-dimensional Banach space with an unconditional Schauder decomposition admits a frequently hypercyclic operator.

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Question 6

Which Banach spaces admit a frequently hypercyclic operator?

Further questions

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For hypercyclicity, the answer is negative (de la Rosa and Read 2009).

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Question 9

Is every chaotic operator frequently hypercyclic?