Two weight norm inequalities for fractional integrals and commutators

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Lecture 2: Two weight inequalities and testing conditions

Two weight norm inequalities

We are interested in conditions on a pair of weights (u, v) such that for $1 < p \leq q < \infty$,

$$\text{Strong } (p,q) \quad I_\alpha : L^p(v) \to L^q(u)$$

$$\left(\int_{\mathbb{R}^n} |I_\alpha f|^q u\, dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f|^p v\, dx\right)^{1/p}$$

$$\text{Weak } (p,q) \quad I_\alpha : L^p(v) \to L^{q,\infty}(u)$$

$$\sup_{t>0} t\, u(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\})^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f|^p v\, dx\right)^{1/p}$$

Also similar inequalities for M_α and $[b, I_\alpha]$

Restating norm inequalities

Let $\sigma = v^{1-p'}$. Then we can restate these inequalities as

$$\text{Strong } (p,q) \quad I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(u)$$

$$\text{Weak } (p,q) \quad I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^{q,\infty}(u)$$

Duality

This formulation has more natural duality:

$$I_\alpha : L^p(v) \to L^p(u) \quad \Leftrightarrow \quad I_\alpha : L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'})$$

$$I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^p(u) \quad \Leftrightarrow \quad I_\alpha(\cdot u) : L^{p'}(u) \to L^{p'}(\sigma)$$

Sawyer testing conditions

- Show operator is bounded if and only if bounded on family of (simple) test functions
- Necessary and sufficient
- Closely related to *Tb* theorems

Nazarov, Treil, Volberg

I_α: Two necessary conditions

Consider the family of test functions $\{\chi_Q\}$:

$$\left(\int_Q I_\alpha(\sigma\chi_Q)^q u\,dx\right)^{1/q} \le M_1 \left(\int_Q \sigma\,dx\right)^{1/p} \qquad (T)$$

and the **dual** inequality

$$\left(\int_Q I_\alpha(u\chi_Q)^{p'} \sigma\,dx\right)^{1/p'} \le M_2 \left(\int_Q u\,dx\right)^{1/q'} \qquad (T^*)$$

The fundamental result

Theorem (Sawyer (1984,1988), LSUT (2009))

Given $1 < p \le q < \infty$ and a pair of weights (u,σ), the testing conditions (T) and (T^*) are necessary and sufficient for the strong type inequality:

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\to L^q(u)} \approx M_1 + M_2.$$

Furthermore, the dual condition (T^*) is necessary and sufficient for the weak type inequality:

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(u)} \approx M_2.$$

Relating weak and strong type

Corollary

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\to L^q(u)} \approx \|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(u)} + \|I_\alpha(\cdot\sigma)\|_{L^{q'}(u)\to L^{p',\infty}(\sigma)}$$

A model result

Theorem (Sawyer (1982))

Given $1 < p \leq q < \infty$ and a pair of weights (u,σ), a necessary and sufficient condition for

$$M_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(u)$$

is that for every cube Q,

$$\left(\int_Q M_\alpha(\sigma\chi_Q)^q u\,dx\right)^{1/q} \leq K\left(\int_Q \sigma\,dx\right)^{1/p} \qquad (MT)$$

Key steps in proof

- Reduce to dyadic case and prove for $L_\alpha^{\mathcal{S}}$, for $\mathcal{S} \subset \mathcal{D}$ sparse.
- Restrict $L_\alpha^{\mathcal{S}}$ to cubes in some $Q_0 \in \mathcal{D}$.
- Form a corona decomposition: intuitively, a weighted Calderón-Zygmund decomposition.
- Rearrange the sum using the corona decomposition and apply the testing condition.

Idea for proof from Hytönen (2012) and LSUT (2012). See also Kairema (2012).

The corona decomposition

Given a function f, a weight σ, define $\mathcal{F} \subset \mathcal{S}$ inductively:

- $\mathcal{F}_0 = \{Q_0\}$
- Given $F \in \mathcal{F}_k$, let $\eta_{\mathcal{F}}(F)$ be maximal cubes $Q \in \mathcal{S}$ such that $Q \subset F$ and $\langle f\rangle_{\sigma,Q} > 2\langle f\rangle_{\sigma,F}$.
- $F_{k+1} = \bigcup_{F\in\mathcal{F}_k} \eta_{\mathcal{F}}(F)$, $\quad \mathcal{F} = \bigcup_k \mathcal{F}_k$.
- Given $Q \in \mathcal{S}$, let $\pi_{\mathcal{F}}(Q)$ be the smallest cube in $\mathcal{F}$ such that $Q \subset \pi_{\mathcal{F}}(Q)$

$\pi_{\mathcal{F}}(Q)$ = "padres" $\quad \eta_{\mathcal{F}}(F)$ = "hijos"

σ-Sparseness

Given $F \in \mathcal{F}$:

$$\sum_{F' \in \eta_{\mathcal{F}}(F)} \sigma(F') \leq \sum_{F' \in \eta_{\mathcal{F}}(F)} \frac{(f\sigma)(F')}{2\langle f\rangle_{\sigma,F}} \leq \frac{(f\sigma)(F)}{2\langle f\rangle_{\sigma,F}} = \frac{1}{2}\sigma(F).$$

Therefore, if,

$$E_{\mathcal{F}}(F) = F \setminus \bigcup_{F' \in \eta_{\mathcal{F}}(F')} F',$$

$$\sigma(E_{\mathcal{F}}(F)) \geq \frac{1}{2}\sigma(F).$$

The sum to estimate ($p = q$)

By duality, we need to show that for every $g \in L^{p'}(u)$

$$\|L_\alpha^{\mathcal{S}}(f\sigma)\|_{L^p(u)} = \int_{\mathbb{R}^n} L_\alpha^{\mathcal{S}}(f\sigma)\, gu\, dx$$
$$= \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} \langle f\sigma\rangle_Q \int_{E(Q)} gu\, dx \leq \|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(u)}.$$

Here assume sum is over $Q \subset Q_0$

The good function

$$\sum_{\substack{Q\in\mathcal{S}\\ Q\subset Q_0}} |Q|^{\frac{\alpha}{n}} \langle f\sigma\rangle_Q \int_{E(Q)} gu\, dx = \sum_{F\in\mathcal{F}} \sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}}(Q)=F}} |Q|^{\frac{\alpha}{n}} \langle f\sigma\rangle_Q \int_{E(Q)} gu\, dx$$

If $\pi_{\mathcal{F}}(Q) = F$, and $F' \in \eta_{\mathcal{F}}(F)$, then $F' \subsetneq Q$.

$$\int_{E(Q)} gu\, dx = \int_{E(Q)\cap E_{\mathcal{F}}(F)} gu\, dx + \sum_{F'\in\eta_{\mathcal{F}}(F)} \int_{E(Q)\cap F'} gu\, dx$$
$$= \int_{E(Q)\cap E_{\mathcal{F}}(F)} gu\, dx = \int_{E(Q)} g_F u\, dx.$$

Applying the testing condition

$$\sum_{F\in\mathcal{F}} \sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}}(Q)=F}} |Q|^{\frac{\alpha}{n}} \langle f\rangle_{\sigma,Q}\langle \sigma\rangle_Q \int_{E(Q)} g_F u\, dx$$
$$\leq \sum_{F\in\mathcal{F}} 2\langle f\rangle_{\sigma,F} \sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}}(Q)=F}} |Q|^{\frac{\alpha}{n}} \langle \sigma\rangle_Q \int_{E(Q)} g_F u\, dx$$
$$\leq \sum_{F\in\mathcal{F}} 2\langle f\rangle_{\sigma,F} \int_F L_\alpha^{\mathcal{S}}(\sigma\chi_F)\, g_F u\, dx$$
$$\leq \sum_{F\in\mathcal{F}} 2\langle f\rangle_{\sigma,F} \|M_\alpha^{\mathcal{D}}(\sigma\chi_F)\chi_F\|_{L^p(u)}\|g_F\|_{L^{p'}(u)}$$
$$\lesssim \sum_{F\in\mathcal{F}} \langle f\rangle_{\sigma,F}\, \sigma(F)^{1/p}\, \|g_F\|_{L^{p'}(u)}.$$

The final estimate

$$\sum_{F\in\mathcal{F}}\langle f\rangle_{\sigma,F}\,\sigma(F)^{1/p}\,\|g_F\|_{L^{p'}(u)}$$

$$\le\left(\sum_{F\in\mathcal{F}}\langle f\rangle_{\sigma,F}^{p}\,\sigma(F)\right)^{1/p}\left(\sum_{F\in\mathcal{F}}\|g_F\|_{L^{p'}(u)}^{p'}\right)^{1/p'}$$

$$\lesssim\left(\sum_{F\in\mathcal{F}}\int_{E_{\mathcal{F}}(F)}M_\sigma^{\mathcal{D}}(f)^p\sigma\,dx\right)^{1/p}\left(\sum_{F\in\mathcal{F}}\int_{E_{\mathcal{F}}(F)}g^{p'}u\,dx\right)^{1/p'}$$

$$\le\left(\int_{\mathbb{R}^n}M_\sigma^{\mathcal{D}}(f)^p\sigma\,dx\right)^{1/p}\left(\int_{\mathbb{R}^n}g^{p'}u\,dx\right)^{1/p'}$$

$$\lesssim\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(u)}$$

Proof fails for I_α

Recall: $I_\alpha^{\mathcal{S}}(f\sigma)(x)=\sum_{Q\in\mathcal{S}}|Q|^{\frac{\alpha}{n}}\langle f\sigma\rangle_Q\chi_Q(x).$

Then for $Q\subset F$

$$\int_Q gu\,dx=\int_{Q\cap E_{\mathcal{F}}(F)}gu\,dx+\sum_{F'\in\eta_{\mathcal{F}}(F)}\int_{Q\cap F'}gu\,dx$$

$$=\int_Q\left(g\chi_{E_{\mathcal{F}}(F)}+\sum_{F'\in\eta_{\mathcal{F}}(F)}\langle g\rangle_{u,F'}\right)u\,dx=\int_Q g_F u\,dx,$$

and we cannot bound $\sum_{F\in\mathcal{F}}\|g_F\|_{L^{p'}(u)}$.

Parallel corona decomposition

- Form second corona decomposition $\mathcal{G}$ of g with respect to u.
- Divide sum into two pieces:

$$\sum_{Q\in\mathcal{S}}=\sum_{\substack{F\in\mathcal{F}\\G\in\mathcal{G}}}\sum_{\substack{\pi_{\mathcal{F}}(Q)=F\\\pi_{\mathcal{G}}(Q)=G}}=\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\G\subset F}}\sum_{\substack{\pi_{\mathcal{F}}(Q)=F\\\pi_{\mathcal{G}}(Q)=G}}+\sum_{G\in\mathcal{G}}\sum_{\substack{F\in\mathcal{F}\\F\subsetneq G}}\sum_{\substack{\pi_{\mathcal{F}}(Q)=F\\\pi_{\mathcal{G}}(Q)=G}}$$

- Estimate first sum as before; use $\mathcal{G}$ to estimate g_F.
- Estimate second sum exchanging roles of (f,σ) and (g,u) and use dual testing condition.

Off-diagonal testing

Theorem (LSUT (2009))

If $1<p<q<\infty$, the strong and weak type inequalities for $I_\alpha^{\mathcal{D}}$ are characterized by

$$\left(\int_Q I_{\alpha,Q}^{\mathcal{D},+}(\sigma\chi_Q)^q u\,dx\right)^{1/q}\le M_1\left(\int_Q\sigma\,dx\right)^{1/p}\qquad(T_+)$$

$$\left(\int_Q I_{\alpha,Q}^{\mathcal{D},+}(u\chi_Q)^{p'}\sigma\,dx\right)^{1/p'}\le M_2\left(\int_Q u\,dx\right)^{1/q'}\qquad(T_+^*)$$

where

$$I_{\alpha,Q}^{\mathcal{D},+}f(x)=\sum_{\substack{Q'\in\mathcal{D}\\Q\subsetneq Q'}}|Q'|^{\frac{\alpha}{n}}\langle f\rangle_{Q'}\,\chi_{Q'}(x).$$

Two conjectures

Conjecture

The two testing conditions

$$\left(\int_Q [b, I_\alpha](\sigma\chi_Q)^q u\,dx\right)^{1/q} \le M_1 \left(\int_Q \sigma\,dx\right)^{1/p} \qquad (CT)$$

$$\left(\int_Q [b, I_\alpha](u\chi_Q)^{p'} \sigma\,dx\right)^{1/p'} \le M_2 \left(\int_Q u\,dx\right)^{1/q'} \qquad (CT^*)$$

are necessary and sufficient for $[b, I_\alpha] : L^p(\sigma) \to L^q(u)$.

(CT^*) *is necessary and sufficient for* $L^p(\sigma) \to L^{q,\infty}(u)$.

Technical obstacles

- Necessity by duality.
- Pass to dyadic operator: how to prove equivalence?
- How to modify parallel corona decomposition to "pull out" $\langle f\rangle_{Q,\sigma}$?

End of Lecture 2
Questions?