Composition operators on Hardy spaces

Episode II

VI Curso Internacional de Análisis Matemático en Andalucía

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Program

2. Lecture 2

- H^∞
- Hardy-Orlicz spaces and their composition operators
- Carleson versus Nevanlinna

Notations

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LLQR=D. Li + H. Queffélec + L. Rodríguez-Piazza + P.L.

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The same holds replacing H^∞ by $A(\mathbb{D})$ (once $\varphi \in A(\mathbb{D})$)

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Contradiction !

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We even know the distance of C_φ to many operator ideals $\mathcal{I}$:

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(L. '09)

Assuming that $\mathcal{K}(H^\infty) \subset \mathcal{I} \subset \mathcal{W}(H^\infty)$, then

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Shall we emphasize new phenomena ?

There, th behavior of C_φ looks rather like on H^2 ? or rather like on H^∞ ?

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We define the Orlicz space $L^\Psi(\mathbb{T})$ as formed by the (classes of) measurable functions $f\colon \mathbb{T} \to \mathbb{C}$ such that

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Point out that a measurable f belongs to the unit ball of L^Ψ *if and only if*

$$\int_{\mathbb{T}} \Psi(|f|)\, d\lambda \leq 1\,.$$

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Actually

$L^\Psi(\mathbb{T}) = M^\Psi(\mathbb{T})$ *if and only if* Ψ satisfies the so-called Δ_2 condition, that is:

$$\limsup_{x\to+\infty} \frac{\Psi(2x)}{\Psi(x)} < +\infty.$$

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- Φ satisfies the Δ_2 condition, then $\left(M^{\Psi}(\mathbb{T})\right)^{**} = L^{\Psi}(\mathbb{T})$.

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In M^Ψ there exists a Dominated Convergence Theorem, which is clearly not true in L^Ψ.

Dominated Convergence Theorem in M^Ψ.

Let $\{f_n\}_n$ be a sequence of measurable functions converging pointwise *a.e.* to f. If there exists $g \in M^\Psi(\mathbb{T})$ such that $|f_n| \leq g$ *a.e.*, for every n, then

$$f_n \xrightarrow{\|\cdot\|_\Psi} f\ .$$

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In fact

$$H^\Psi = \{f \in H^1 : f^* \in L^\Psi(\mathbb{T})\}$$

Big Orlicz norms

If $1 < p < r < +\infty$, and if Ψ satisfies the condition Δ^2:

$$\exists\, A > 1 \text{ and } x_0 > 0 \ s.t. \quad \Psi(Ax) \geq \big(\Psi(x)\big)^2, \qquad \text{for all } x \geq x_0,$$

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then

$$H^1 \supset H^p \supset H^r \supset H^\Psi \supset H^\infty$$

and

$$\|\cdot\|_1 \leq \|\cdot\|_p \leq \|\cdot\|_r \lesssim \|\cdot\|_\Psi \lesssim \|\cdot\|_\infty$$

Examples: $\Psi(x) = \Psi_q(x) = e^{x^q} - 1$.

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So $\|u_{a,r}\|_{H^\Psi} \leq 1/\Psi^{-1}(\frac{1}{1-r})$. Thanks to $(**)$, we actually have

$$\|u_{a,r}\|_{H^\Psi} \approx \frac{1}{\Psi^{-1}\big(1/(1-r)\big)}\,.$$

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We know $P_z \geq 0$, $\|P_z\|_1 = 1$, $\|P_z\|_\infty = \dfrac{1+|z|}{1-|z|}$, and, for $f \in B_{H^\Psi}$,

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Actually, for many Ψ, we have: $\quad \|\delta_z\|_{(H^\Psi)^*} \leq \Psi^{-1}\Big(\dfrac{1}{1-|z|^2}\Big)$.

Boundedness on Hardy–Orlicz spaces

For every $\varphi\colon \mathbb{D} \to \mathbb{D}$ and any Orlicz function Ψ, the operator $C_\varphi\colon H^\Psi \to H^\Psi$ is (well defined and) bounded .

Assuming $\varphi(0) = 0$, we use again the Littlewood's Subordination Principle:

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For general φ, once again we write $C_\varphi = C_\phi \circ C_{q_a}$ where $\phi(0) = 0$.

Compactness on Hardy–Orlicz spaces

The Schwartz's criterium for compactness is still valid.

The composition operator $C_\varphi \colon H^\Psi \to H^\Psi$ is compact

if and only if

for every bounded sequence $\{f_n\}_n$ in H^Ψ converging to 0 uniformly on compact subsets of $\mathbb{D}$, we have $f_n \circ \varphi \to 0$ in H^Ψ.

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Corollary

If $C_\varphi \colon H^\Psi \to H^\Psi$ is compact, then

$$\lambda_\varphi(\mathbb{T}) = 0 \quad \textit{i.e.} \quad |\varphi^*| < 1 \textit{ almost everywhere on } \mathbb{T}.$$

Compactness on Hardy–Orlicz spaces

A second consequence is as follows. Remember

$$u_{a,r}(z) = \left(\frac{1-r}{1-\bar{a}rz}\right)^2 \quad \text{and} \quad \|u_{a,r}\|_{H^\Psi} \approx \frac{1}{\Psi^{-1}\big(1/(1-r)\big)}\,.$$

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Corollary

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   $\exists g \in M^\Psi(\mathbb{T})$, with $g \geq |(C_\varphi f)^*|$ a.e., for all $f \in B_{H^\Psi}$.

Recall: Order bounded in L^2 is the same that Hilbert–Schmidt...

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Remark: for all the functions Ψ_q, this is equivalent to the (simple) condition:

$$\forall d \geq 1\,, \quad \int_{\mathbb{T}} |\varphi|^n \, d\lambda = o\left(n^{-d}\right)$$

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Observe that the best g we can choose to majorize $|(C_\varphi f)^*|$, for every f in the unit ball is

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Fix $B > 1$. Let $\varepsilon = \varepsilon(B) > 0$ (specified later), there exists $r_0 < 1$ s.t., for every $a \in \mathbb{T}$, and all $r_0 < r < 1$:

$$\int_{\overline{\mathbb{D}}} \Psi\left(\frac{1}{\varepsilon}\Psi^{-1}\left(\frac{1}{1-r}\right)\left|u_{a,r}(z)\right|\right) d\lambda_\varphi = \int_{\mathbb{T}} \Psi\left(\frac{1}{\varepsilon}\Psi^{-1}\left(\frac{1}{1-r}\right)\left|u_{a,r}\circ\varphi^*\right|\right) d\lambda \leq 1\,.$$

Order bounded composition operators

It remains to prove that, under condition (Δ^2), compactness implies order boundedness.
This will be done using the condition

$$\lim_{r\to 1^-}\sup_{a\in\mathbb{T}}\Psi^{-1}\left(\frac{1}{1-r}\right)\left\|C_\varphi(u_{a,r})\right\|_{H^\Psi}=0 \qquad (W)$$

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Using Markov's inequality and the fact that $|u_{a,r}(z)|\ge 1/4$, whenever $|z-a|\le 1-r$,

$$1\ge\lambda(\{|\varphi^*-a|\le 1-r\})\Psi\left(\frac{1}{4\varepsilon}\Psi^{-1}\left(\frac{1}{1-r}\right)\right),\quad \text{for every } a\in\mathbb{T}.$$

Order bounded composition operators

Observe that the annulus $\{z \in \mathbb{C} : 1 - h < |z| < 1\}$, for h small enough, can be covered by less than C/h balls of radii $2h$ and centers in $\mathbb{T}$ and therefore, taking $2h = 1 - r$

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$$C\Psi(x) \geq \lambda(\{g > x\}) \cdot \Psi(x/8\varepsilon) \geq \lambda(\{g > x\})\left[\Psi(x/8A^2\varepsilon)\right]^4 = \lambda(\{g > x\})\left[\Psi(Bx)\right]^4$$

using condition (Δ^2) twice and for a suitable choice of ε.

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Namely $g \in M^\Psi(\mathbb{T})$ and

$$C_\varphi \text{ is order bounded in } M^\Psi.$$

ok !

A question

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We shall answer a bit later....suspense...

Back to pullback measure

We already mentioned (recall lecture 1) that, when $f \in H^p$, we have

$$\|C_\varphi f\|_{H^p}^p = \|(f \circ \varphi)^*\|_{L^p(\mathbb{T})}^p = \int_{\mathbb{T}} |f|^p \circ \varphi^* \, d\lambda = \|f\|_{L^p(\lambda_\varphi)}^p$$

where λ_φ to the pullback measure of λ by the map φ^*: $\lambda_\varphi(B) = \lambda(\{\varphi^* \in B\})$, for every Borel set $B \subset \overline{\mathbb{D}}$.

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So properties like boundedness, compactness,... of the operator C_φ are the same than the properties of the inclusion (embedding) operator

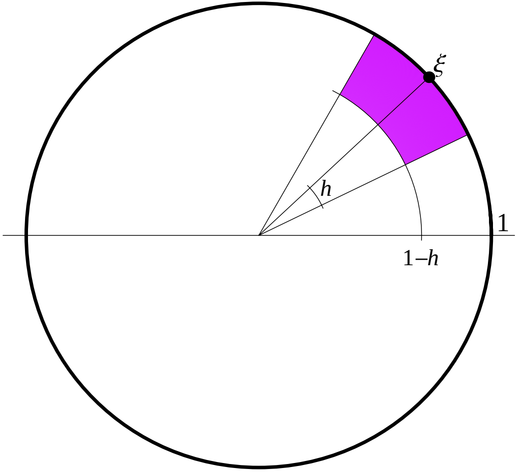
$$j_{\lambda_\varphi} : H^p \hookrightarrow L^p(\lambda_\varphi)\,.$$

The same argument works for Hardy–Orlicz spaces.

Carleson windows

Let $0 < h < 1$. We define the window of center $\xi \in \mathbb{T}$ and radius h as

$$W(\xi, h) = \{z \in \overline{\mathbb{D}} : 1 - h < |z|, |\arg(\bar{\xi} z)| < h\}$$



Carleson's point of view

Let $1 \leq p < \infty$. Let μ be a finite measure on the borelian sets of $\overline{\mathbb{D}}$.
Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic.

Carleson, '62

The identity $H^p \longrightarrow L^p(\mu)$ defines a bounded operator *if and only if* μ is a Carleson measure:

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The pullback measures λ_φ are Carleson measures.

An example

Let us recall the proposed question:

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If we have $\varphi_1, \varphi_2 : \mathbb{D} \to \mathbb{D}$ holomorphic, such that $|\varphi_1^*| = |\varphi_2^*|$ a.e. on $\mathbb{T}$, and $C_{\varphi_1} : H^2 \to H^2$ is compact.

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We are going to give two holomorphic functions φ and $M \colon \mathbb{D} \longrightarrow \mathbb{D}$, with M an inner function, such that C_φ is not compact, but $C_{M.\varphi}$ is compact.

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Taking $\varphi_2 = \varphi$, and $\varphi_1 = M.\varphi$, we see that the answer to the proposed problem is NO.

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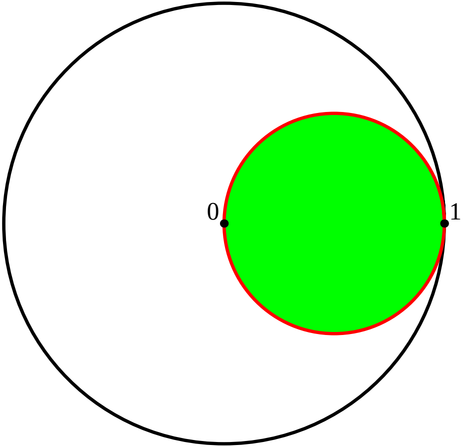

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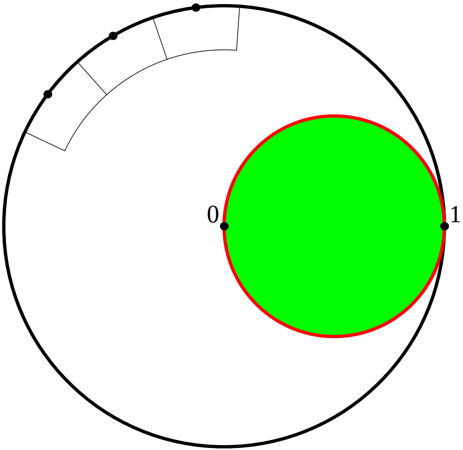

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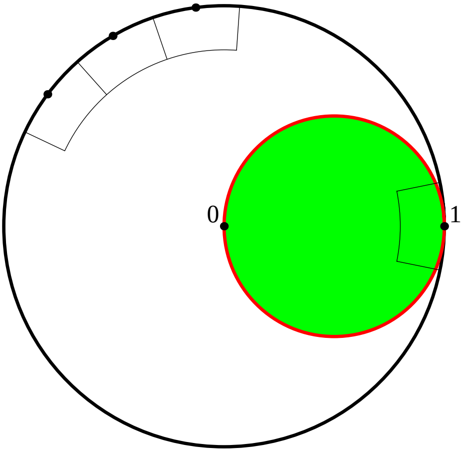

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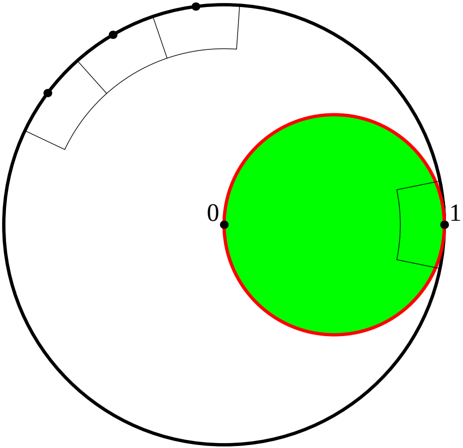

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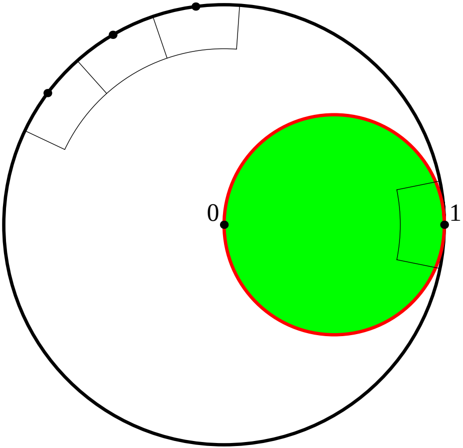

We have $\rho_\varphi(h) \approx h \quad \Longrightarrow \quad C_\varphi$ is not compact.

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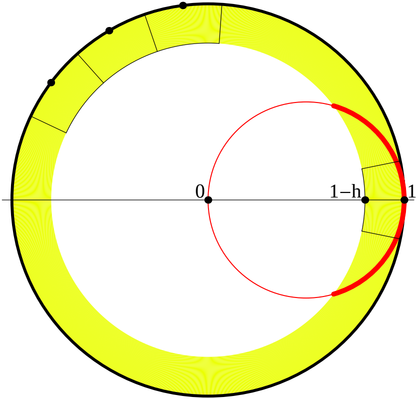

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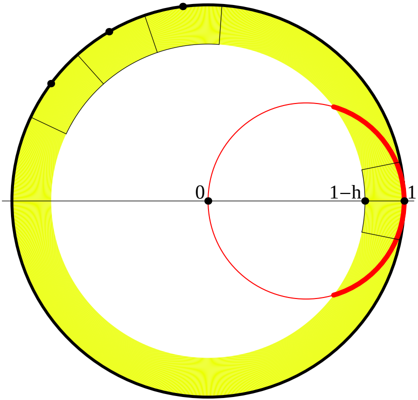

If M is an inner function, we also have $\lambda_{\varphi M}\big(\{z \in \overline{\mathbb{D}} : |z| > 1 - h\}\big) \approx \sqrt{h}$.

An example

We choose $M(z) = \exp\big(T(z)\big)$, where $T(z) = \dfrac{z+1}{z-1}$.
Point out that T is a conformal mapping from the disk $\mathbb{D}$ to the left half plane.
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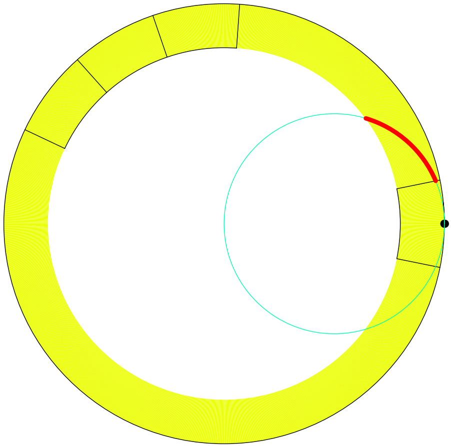

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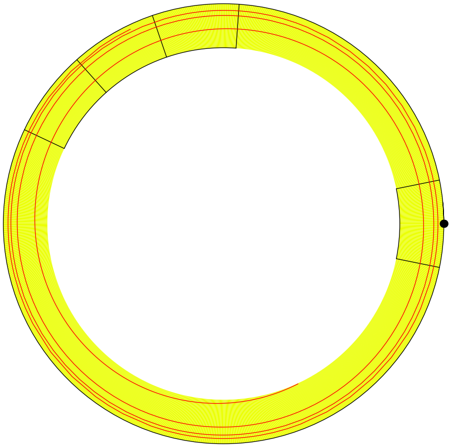

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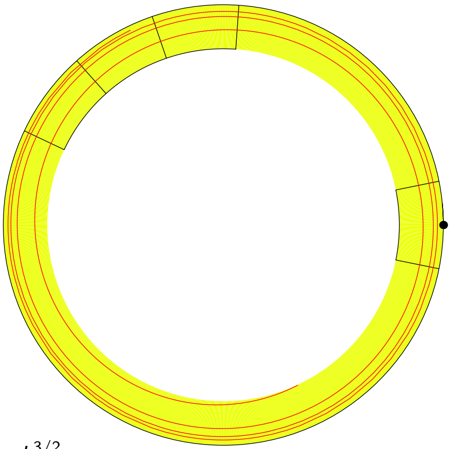

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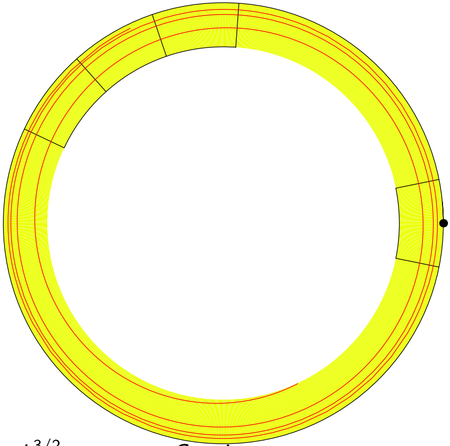

We have $\rho_{\varphi M}(h) \approx h^{3/2} \quad \Longrightarrow \quad C_{\varphi M}$ is compact.

Compactness of the inclusion in $L^\Psi(\mu)$

The compactness of composition operator $C_\varphi\colon H^\Psi \to H^\Psi$ is equivalent to the compactness of the inclusion operator of H^Ψ in $L^\Psi(\lambda_\varphi)$.

So we could try to characterize for which finite measure μ on $\mathbb{D}$, is the inclusion operator $H^\Psi \hookrightarrow L^\Psi(\mu)$ compact.

Compactness of the inclusion in $L^\Psi(\mu)$

Let μ be a finite measure on $\overline{\mathbb{D}}$, $h \in (0,1)$ and $A > 0$.
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Theorem ('07)

Consider the following conditions:

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(C_0) The inclusion of $H^\Psi(\mathbb{D})$ in $L^\Psi(\mu)$ is a compact operator.

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(K_0) For every $A > 0$, $\quad K_\mu(h) = o\left(\frac{\gamma_A(h)}{h}\right), \quad h \to 0^+$.

(C_0) The inclusion of $H^\Psi(\mathbb{D})$ in $L^\Psi(\mu)$ is a compact operator.

Then we have $\quad (K_0) \Longrightarrow (C_0) \Longrightarrow (R_0)$.

Compactness of composition operators on H^Ψ

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Regularity of the pullback measure ('07)

There exists a constant $k_1 > 0$ so that, for every analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$, and for every $\xi \in \mathbb{T}$, we have

$$\lambda_\varphi\big(W(\xi, \varepsilon h)\big) \le k_1\, \varepsilon\, \lambda_\varphi\big(W(\xi, h)\big)\,,$$

whenever $0 < \varepsilon < 1$, and $0 < h < 1 - |\varphi(0)|$.

Compactness of composition operators on H^Ψ

Consequence: For $\mu = \lambda_\varphi$ we have

$$K_\mu(h) = \sup_{0<t\leq h} \frac{\rho_\mu(t)}{t} \approx \frac{\rho_\mu(h)}{h}$$

Therefore

$$(R_0) \quad \rho_\mu(h) = o\big(\gamma_A(h)\big), \quad h \to 0^+ \text{ and } (K_0) \quad K_\mu(h) = o\Big(\frac{\gamma_A(h)}{h}\Big), \quad h \to 0^+$$

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Characterization of compactness (LLQR '07)

The composition operator $C_\varphi \colon H^\Psi \to H^\Psi$ is compact *if and only if*

$$\forall A > 0\,, \qquad \rho_{\lambda_\varphi}(h) = o\Big(\gamma_A(h)\Big), \quad \text{when } h \to 0^+.$$

if and only if

$$\lim_{h\to 0^+} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}\big(1/\rho_{\lambda_\varphi}(h)\big)} = 0.$$

Some consequences

- If C_φ is compact on H^Ψ, then it is compact on H^2.

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- When Ψ satisfies the Δ_2 condition, C_φ is compact on H^Ψ if and only if it is compact on H^2.
- Conversely, if Ψ does not satisfy the Δ_2 condition, there exists a symbol φ such that C_φ is compact on H^2, but it is not compact on H^Ψ.

Some consequences

- If Ψ satisfies, for some $\beta > 1$,

$$\lim_{x \to +\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty$$

then C_φ is compact on H^Ψ *if and only if* it is weakly compact.

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  Moreover φ can be onto ($\varphi(\mathbb{D}) = \mathbb{D}$).
  This is done building an example similar to the one of Shapiro and MacCluer; changing their set Ω for a new set Ω_1, adapted to Ψ.

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The Nevanlinna counting function is "equivalent" to the λ_φ-measure of the Carleson's windows.

Carleson versus Nevanlinna

(LLQR '11)

There exist $c, C > 0$ (numerical) s.t.

- $N_\varphi(w) \leq C\lambda_\varphi\Big(W(\xi, ch)\Big)$ where $w = \xi(1-h)$
- $\lambda_\varphi(W(\xi, h)) \leq \dfrac{C}{\mathcal{A}(W(\xi, ch))} \displaystyle\int_{W(\xi,ch)} N_\varphi(w)\, d\,\mathcal{A} \leq C \sup_{w \in W(\xi,ch)} N_\varphi(w)$

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- $C_\varphi : H^\Psi \to H^\Psi$ is compact *if and only if* $\displaystyle\lim_{|w|\to 1^-} \frac{\Psi^{-1}\big(1/(1-|w|)\big)}{\Psi^{-1}\big(1/N_\phi(w)\big)} = 0.$

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- When φ is finitely-valent:

$$C_\varphi : H^\Psi \to H^\Psi \text{ is compact } \textit{if and only if } \lim_{|z|\to 1^-} \frac{\Psi^{-1}\Big(\dfrac{1}{1-|\varphi(z)|}\Big)}{\Psi^{-1}\Big(\dfrac{1}{1-|z|}\Big)}$$

Key ingredients of the proof

The main tool is an Orlicz version of the Littlewood-Paley formula:

Stanton formula '85

Let $G\colon \mathbb{D} \longrightarrow \mathbb{R}$ be subharmonic.

$$\lim_{r\to 1^-} \int_{\mathbb{T}} G\big(\varphi(r\xi)\big)\, d\lambda(\xi) = G\big(\varphi(0)\big) + \frac{1}{2}\int_{\mathbb{D}} \Delta G(w) N_\varphi(w)\, dA(w),$$

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For instance to majorize the Nevanlinna function, we concretely choose $s \mapsto \psi''(s)$, as a step function, depending on $\lambda_\varphi\Big(W(\frac{w}{|w|}, c(1-|w|)\Big)$.

Merci !