Convex inequalities, isoperimetry and spectral gap III

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Part III. K-L-S spectral gap conjecture

- KLS estimate, through Milman's theorem
- Gaussian case
- Other related conjectures

Kannan-Lovász-Simonovits problem

Given μ a log-concave probability on $\mathbb{R}^n$, estimate the biggest constants

$$\mu^+(A) \geq C\mu(A) \qquad \forall \text{ borelian, } \mu(A) \leq \frac{1}{2}$$

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We know

$$\Updownarrow$$

$$\lambda_2 \mathbb{E}_\mu |f - \mathbb{E}_\mu f|^2 \leq \mathbb{E}_\mu |\nabla f|^2 \qquad \forall \text{ locally Lipschitz } f$$

$$\Updownarrow$$

$$C' \mathbb{E}_\mu |f - \mathbb{E}_\mu f|^2 \leq 1 \qquad \forall \text{ 1-Lipschitz } f$$

$$C^2 \simeq \lambda_2 \simeq C'^2$$

Kannan-Lovász-Simonovits conjecture

KLS conjecture

The biggest constant is attained for affine functions, up to an absolute constant.

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- f affine, $f(x) = t + \langle a, x\rangle$, $t \in \mathbb{R}$, $a \in \mathbb{R}^n$
- f affine 1-Lipschitz, $f(x) = t + \langle \theta, x\rangle$, $t \in \mathbb{R}$, $\theta \in S^{n-1}$
- $\mathbb{E}_\mu f = t + \langle \theta, \mathbb{E}_\mu x\rangle$
- Then,
$$\mathbb{E}_\mu |f - \mathbb{E}_\mu f|^2 = \mathbb{E}_\mu \langle \theta, x - \mathbb{E}_\mu x\rangle^2$$
- Compute
$$\lambda_\mu^2 := \sup_{\theta \in S^{n-1}} \mathbb{E}_\mu \langle \theta, x - \mathbb{E}_\mu x\rangle^2$$

($\lambda_\mu :=$ highest eigenvalue of the covariance matrix of μ)

Kannan-Lovász-Simonovits conjecture

Let μ a log-concave probability in $\mathbb{R}^n$ with barycenter $\mathbb{E}_\mu x$. Let

$$\lambda_\mu^2 = \sup_{\theta \in S^{n-1}} \mathbb{E}_\mu \langle x - \mathbb{E}_\mu x, \theta \rangle^2$$

then Poincaré inequality is

$$\mathbb{E}_\mu |f - \mathbb{E}_\mu f|^2 \leq C \lambda_\mu^2 E_\mu |\nabla f|^2$$

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- $C > 0$ is an absolute constant, independent of μ and even of the dimension,
- λ_μ^2 is also the highest eigenvalue for the covariance matrix for μ, say

$$A = \mathbb{E}_\mu (x - \mathbb{E}_\mu x) \otimes (x - \mathbb{E}_\mu x) = \left(\mathbb{E}_\mu (x_i x_j - \mathbb{E}_\mu x_i \mathbb{E}_\mu x_j)^2 \right)_{1 \leq i,j \leq n}$$

KLS estimate

Given μ (log-concave) and f Lipschitz

$$\mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 \leq C\mathbb{E}_\mu|x - \mathbb{E}_\mu x|^2 \cdot \mathbb{E}_\mu|\nabla f|^2$$

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Proof.

$$\begin{aligned}
\mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 &\leq \mathbb{E}_\mu\big(|f(x) - f(\mathbb{E}_\mu x)| + |f(\mathbb{E}_\mu x) - \mathbb{E}_\mu f|\big)^2 \\
&= \mathbb{E}_\mu\big(|f(x) - f(\mathbb{E}_\mu x)| + |\mathbb{E}_\mu(f(\mathbb{E}_\mu x) - f)|\big)^2 \\
&\leq \text{ (by Minkowski's inequality)} \\
&\leq 4\,\mathbb{E}_\mu|f(x) - f(\mathbb{E}_\mu x)|^2 \\
&\leq 4\mathbb{E}_\mu|x - \mathbb{E}_\mu x|^2 \cdot \||\nabla f|\|_\infty^2 \\
&\leq C\,\underbrace{\mathbb{E}_\mu|x - \mathbb{E}_\mu x|^2 \cdot \mathbb{E}_\mu|\nabla f|^2 \qquad \big(\leq C\,n}\,\lambda_\mu^2 \cdot \mathbb{E}_\mu|\nabla f|^2\big)
\end{aligned}$$

$$\mathbb{E}_\mu|x - \mathbb{E}_\mu x|^2 = \textstyle\sum_{i=1}^n \mathbb{E}_\mu(x_i - \mathbb{E}_\mu x_i)^2$$

Previous results

Payne-Weinberger (1960)

If μ is the normalized measure in a convex body K

$$\mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 \leq \frac{4}{\pi^2}\,\mathrm{diam}(K)^2 \cdot \mathbb{E}_\mu|\nabla f|^2$$

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If B Euclidean ball and μ its normalized measure

$$\mathbb{E}_\mu |f - \mathbb{E}_\mu f|^2 \leq \frac{C}{n} \cdot \mathbb{E}_\mu |\nabla f|^2$$

(the right estimate)

Examples of product spaces

Talagrand (1991)

Let $d\mu(x) = \frac{1}{2^n} e^{-\sum_{i=1}^n |x_i|} dx$. Then

$$\mathbb{E}_\mu |f - \mathbb{E}_\mu f|^2 \leq C \cdot \mathbb{E}_\mu |\nabla f|^2$$

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Gaussian case

Let $d\mu(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$. Then

$$\mathrm{Var}_\mu f = \mathbb{E}_\mu |f - \mathbb{E}_\mu f|^2 \leq \mathbb{E}_\mu |\nabla f|^2$$

Proof of Gaussian case

It is easy to see that

$$\lambda_\mu = 1$$

Let now $u \in \mathcal{D}(\mathbb{R}^n)$, test functions.
Consider the associated Laplace Beltrami operator L

$$Lu(x) = \Delta u(x) - \langle x, \nabla u(x) \rangle$$

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We know:

- $\{Lu; u \in \mathcal{D}\}$ is dense in $\{f \in L^2(\mathbb{R}^n, d\mu) : \mathbb{E}_\mu f = 0\}$
- Then, $\inf_{u \in \mathcal{D}} \left\{ \mathbb{E}_\mu (Lu - f)^2 \right\} = 0$

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and integrating by parts

- $\mathbb{E}_\mu f \, Lu = -\mathbb{E}_\mu \langle \nabla f, \nabla u \rangle$ (Green formula)
- $\mathbb{E}_\mu (Lu)^2 = \mathbb{E}_\mu \langle \nabla u, \nabla u \rangle + \mathbb{E}_\mu \sum_{i,j} (\partial_{ij} u(x))^2 \geq \mathbb{E}_\mu |u|^2$

Assume $\mathbb{E}_\mu f = 0$. Since

$$\mathrm{Var}_\mu f = \mathbb{E}_\mu \big| f - \mathbb{E}_\mu f \big|^2 = \mathbb{E}_\mu f^2$$

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$$\begin{aligned}
\mathbb{E}_\mu f^2 - \mathbb{E}_\mu (Lu - f)^2 &= 2\mathbb{E}_\mu f\, Lu - \mathbb{E}_\mu (Lu)^2 \\
&\le -2\mathbb{E}_\mu \langle \nabla f, \nabla u \rangle - \mathbb{E}_\mu |u|^2 \\
&\le \mathbb{E}_\mu |\nabla f|^2
\end{aligned}$$

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Taking the infimum in u we obtain the result.

$$\mathrm{Var}_\mu f \leq \lambda_\mu^2 \cdot \mathbb{E}_\mu |\nabla f|^2$$

Known results

The normalized measure on

- p-balls, $1 \leq p \leq \infty$ (Sodin 2008 , Łatala&Oleskievicz 2008)
- The simplex (Barthe and Wolff, 2009)
- Some revolution bodies (Bobkov, 2003, Hue)
- Unconditional bodies (Klartag, 2009)with a $\log n$ constant, i.e.

$$\mathrm{Var}_\mu f = \mathbb{E}_\mu \Big| f - \mathbb{E}_\mu f \Big|^2 \leq C \log n \, \lambda_\mu^2 \, \mathbb{E}_\mu |\nabla f|^2$$

K is unconditional: $(x_1, \ldots, x_n) \in K$ iff $(|x_1|, \ldots, |x_n|) \in K$

Better known estimate

Guedon-Milman (2011) + Eldan (2013)

Poincaré inequality is true in the following way

$$\mathrm{Var}_\mu f = \mathbb{E}_\mu \Big| f - \mathbb{E}_\mu f \Big|^2 \leq C n^{2/3} (\log n)^2 \, \lambda_\mu^2 \, \mathbb{E}_\mu |\nabla f|^2$$

for any locally Lipschitz integrable function f.

Geometric connections: Three conjectures

- The slicing problem ($\sim$ 1986, Bourgain)

- Thin shell width conjecture (2003, Bobkov-Koldobsky, Antilla-Ball-Perissinaki)

- Kannan-Lovász-Simonovits spectral gap conjecture (1995)

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$\Downarrow$ (Ball-Nguyen 2013)

- The slicing problem

The slicing problem

Conjecture (Bourgain, 1986)

There exists an absolute constant $C > 0$ such that every convex body K in $\mathbb{R}^n$ with volume 1 has, at least, one $(n-1)$-dimensional section such that

$$|K \cap H|_{n-1} \geq C$$

(Optimistic estimate $C = \frac{1}{e}$!)

Slicing problem is true in

- Unconditional convex bodies
- Zonoids
- Random polytopes
- Polytopes in which the number of vertices is proportional to de dimension, i.e., for instance, $N/n \leq 2$
- The unit balls of finite dimensional Schatten classes,for $1 \leq p \leq \infty$
- $(n-1)$-orthogonal projection of the classes above
- and more

Known general estimates

- Bourgain (1986), $|K \cap H|_{n-1} \geq \dfrac{C}{n^{1/4} \log n}$
- Klartag (2006), $|K \cap H|_{n-1} \geq \dfrac{C}{n^{1/4}}$

Thin shell width conjecture (2003, Bobkov-Koldobsky, Antilla-Ball-Perissinaki)

Conjecture

There exists an absolute constant $C > 0$ such that for every log-concave probability μ in $\mathbb{R}^n$

$$\sigma_\mu = \sqrt{\mathbb{E}_\mu \big| |x| - \mathbb{E}_\mu |x| \big|^2} \leq C \lambda_\mu.$$

Remark:

- It is equivalent to KLS conjecture to be true only for the function $|x|$ or $|x|^2$

The name is due to the following fact

If the thin shell width conjecture were true, we would have a stronger concentration of the mass around the mean for log-concave probabilities

$$\mu\left\{\big||x|-\mathbb{E}_\mu|x|\big|>t\mathbb{E}_\mu|x|\right\}\leq 2\exp\left(-C' t^{\frac{1}{2}}\frac{(\mathbb{E}_\mu|x|)^{\frac{1}{2}}}{\lambda_\mu^{\frac{1}{2}}}\right), \qquad \forall\, t>0$$

Thin shell width conjecture is true for

The uniform probability on

- Finite dimensional p-balls, $1 \leq p \leq \infty$
- Finite dimensional Orlicz -balls
- Revolution bodies
- $(n-1)$-dimensional orthogonal projections of the crosspolytope (1-ball)
- $(n-1)$-dimensional orthogonal projections of the cube and even all their linear deformations
- Remark: This conjecture is not linear invariant, but in a random sense, more than half of linear deformations of the classe above also satisfy this conjecture

Best known estimate Guedon-Milman, 2010

There exists an absolute constant $C > 0$ such that for every log-concave probability μ in $\mathbb{R}^n$

$$\sigma_\mu \leq C n^{1/3} \lambda_\mu.$$

Our last contribution in the thin shell width conjecture, joint work with D. Alonso (2013 + ?)

- $(n-1)$-dimensional orthogonal projections of the crosspolytope (1-ball) satisfy this conjecture
- $(n-1)$-dimensional orthogonal projections of the cube and even all their linear deformation satisfy this conjecture
- $(n-1)$-dimensional orthogonal projections of the p-balls satisfy this conjecture whe the vectors on which the projection is taken are sparse.
- If μ satisfies the t.s.w. conjecture then $\nu = \mu \circ T$ also satisfies the t.s.w. conjecture at least for half of T's (T linear mapp) in a probabilistis meaning and 'at random' if Schatten norm of $\|T\|_{c_4}$ satisfies

$$\frac{\|T\|_{HS}}{\|T\|_{c_4}} = o(n^{\frac{1}{4}})$$

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