Decomposition norm theorem, L^p-behavior of reproducing kernels and two weight inequality for Bergman projection.

José Ángel Peláez

Departamento de Análisis Matemático
Universidad de Málaga
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Joint works with O. Constantin and J. Rättyä

VI International Course of Mathematical Analysis in Andalucía
Antequera

Outline of the lecture

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- A little bit on the big picture of weighted Bergman spaces.

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- For $0 < p \le \infty$, H^p, the classical **Hardy space**,

$$||f||_{H^p} = \sup_{0 \le r < 1} M_p(r, f) < \infty.$$

- The Bloch space $\mathcal{B}$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|) + |f(0)| < \infty.$$

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- Standard weighted Bergman spaces, $A^p_\alpha = \mathcal{H}(\mathbb{D}) \cap L^p_\alpha$,
$$\omega(z)dA(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z), \quad -1 < \alpha < \infty.$$

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- A Littlewood-Paley formula

$$||f||^p_{A^p_\alpha} \asymp |f(0)|^p + ||f'||^p_{A^p_{\alpha+p}}.$$

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2. The polynomials are dense in A^p_ω whenever ω is radial, but this is not true in general. Example; $\omega(z) = |S(z)|^2 = \left|\exp\left(-\frac{1+z}{1-z}\right)\right|^2 = \exp\left(-\frac{1-|z|^2}{|1-z|^2}\right)$

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- Primary obstacle; $\{B^\omega_z\}_{z\in\mathbb{D}}$ do not have a neat expression.

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- Exponential type weights

$$\omega(r) = \exp\left(-\frac{C}{(1-r)^\alpha}\right), \quad C,\alpha>0$$

are not doubling.

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- However, techniques which work for A^p_ω, $\omega \in \widehat{\mathcal{D}}$, work for A^p_α and even for H^p (sometimes).

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- Our approach to the two weight inequality for the Bergman projection

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lead us to the consider following equivalent norm.

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- L^p estimates of Bergman reproducing kernels induced by a radial weight

$$B^\omega_a(z)=\sum_{n=0}^\infty \frac{(\bar{a}z)^n}{2\int_0^1 r^{2n+1}\omega(r)\,dr}.$$

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- $$I(0) = I_\omega(0) = \left\{ k \in \mathbb{N} \cup \{0\} : k < E\left(\frac{1}{1-r_1}\right) \right\}$$
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Assume that $\omega \in \widehat{\mathcal{D}}$ with $\int_0^1 \omega(r)\,dr = 1$, and $f \in \mathcal{H}(\mathbb{D})$.

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1. Preliminary result on power series with positive coefficients
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4. Similar results have been obtained by Mateljević and Pavlović for A^p_α.

Proposition (P-Rättyä 2013)

Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$ such that $\int_0^1 \omega(r)\,dr = 1$. Let $f(r) = \sum_{k=0}^\infty a_k r^k$, where $a_k \geq 0$ for all $k \in \mathbb{N} \cup \{0\}$, and denote $t_n = \sum_{k \in I_\omega(n)} a_k$. Then there exists a constant $C = C(p, \omega) > 0$ such that

$$\frac{1}{C} \sum_{n=0}^\infty 2^{-n} t_n^p \leq \int_0^1 f(r)^p \omega(r)\,dr \leq C \sum_{n=0}^\infty 2^{-n} t_n^p.$$

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$$f(r)^p \le \left(\sum_{n=0}^{\infty} t_n r^{M_n}\right)^p \le \eta_\gamma(r)^{p-1} \sum_{n=0}^{\infty} 2^{-n\gamma(p-1)} t_n^p r^{M_n}.$$

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3. This together with a right control of $\eta_\gamma(r)$ and

$$\int_0^1 s^x \omega(s)\, ds \asymp \int_{1-\frac{1}{x}}^1 \omega(s)\, ds = \widehat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1, \infty)$$

finishes the proof.

Theorem (P-Rättyä 2013)

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(Mateljevic-Pavlovic (1984))

$$\|\Delta^\omega_n f\|_{H^p} \asymp M_p\left(r_{n+1}, \Delta^\omega_n f\right)$$

Sketch of the proof. Case $p > 1$.

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\begin{aligned}
\|f\|^p_{A^p_\omega} &\geq \sum_{n=0}^{\infty} \int_{r_{n+1}}^{r_{n+2}} M_p^p(r,f) r\omega(r)\,dr \\
&\gtrsim \sum_{n=0}^{\infty} \int_{r_{n+1}}^{r_{n+2}} M_p^p(r,\Delta_n^\omega f) r\omega(r)\,dr \quad \text{Riesz projection theorem} \\
&\gtrsim \sum_{n=0}^{\infty} \|\Delta_n^\omega f\|^p_{H^p} \int_{r_{n+1}}^{r_{n+2}} r^{pM_{n+1}} \omega(r)\,dr \quad \text{previous estimate} \\
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$$
M_p(r,f) \le \sum_{n=0}^\infty M_p(r,\Delta_n^\omega f) \le \sum_{n=0}^\infty r^{M_n}\|\Delta_n^\omega f\|_{H^p},
$$

and hence

$$
\|f\|^p_{A^p_\omega} \le \int_0^1 \left(\sum_{n=0}^\infty r^{M_n}\|\Delta_n^\omega f\|_{H^p}\right)^p \omega(r)\,dr \asymp \sum_{n=0}^\infty 2^{-n}\|\Delta_n^\omega f\|^p_{H^p}.
$$