Composition operators on Hardy spaces

Episode III

VI Curso Internacional de Análisis Matemático en Andalucía

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Program

3. Other operator ideals

- Schatten classes and approximation numbers.
- Absolutely summing composition operators (work in progress, with L. Rodríguez-Piazza).
- Some open questions...

Schatten Classes

Definition

Let H be a (separable) Hilbert spaces, and T a bounded operator on H.
For $p \geq 1$, define the Schatten p-norm of T as

$$\|T\|_{\mathcal{S}^p} := \left(\sum_{n \geq 1} \lambda_n^p(|T|) \right)^{1/p} = \left(tr(|T|^p) \right)^{1/p}$$

where
$\lambda_1(|T|) \geq \lambda_2(|T|) \geq \cdots \geq \lambda_n(|T|) \geq \cdots$ are the eigenvalues of $|T| = \sqrt{(T^*T)}$.

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Remark: T belongs to $\mathcal{S}^2$ *if and only if* T is Hilbert-Schmidt.

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$$R_{n,j} = \Big\{ z \in \mathbb{D}\,;\; 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \quad \text{and} \quad \frac{2j\pi}{2^n} \leq \arg z < \frac{2(j+1)\pi}{2^n} \Big\}$$

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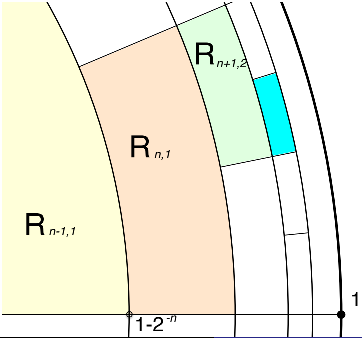

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(Luecking '87)

We assume that $\lambda_\varphi(\mathbb{T}) = 0$.

$$C_\varphi \in \mathcal{S}_p \qquad \textit{if and only if} \qquad \sum_{n\geq 0} \sum_{j=0}^{2^n-1} \left[2^n \lambda_\varphi(R_{n,j})\right]^{p/2} < +\infty.$$

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Link with Carleson's measures ? With α-Carleson ?

A finite measure μ on $\mathbb{D}$ is α-Carleson if $\rho_\mu(h) = \sup_{\xi\in\mathbb{T}} \mu\big(W(\xi,h)\big) = O\big(h^\alpha\big)$.

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A necessary condition

If $C_\varphi \in \mathcal{S}_p$, then $\rho_\varphi(h) = o\left(h\left(\log \frac{1}{h}\right)^{-2/p}\right)$.

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If λ_φ is α-Carleson where $\alpha > 1$, then $C_\varphi \in \mathcal{S}_p$ for any $p > \dfrac{2}{\alpha - 1}$.

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$\forall \alpha \in (1,2)$, there exist two symbols φ_1 and φ_2 such that $|\varphi_1^*| = |\varphi_2^*|$ $(a.e.)$, with

$$\rho_{\varphi_1}(h) \approx h \qquad \text{et} \qquad \rho_{\varphi_2}(h) \approx h^\alpha$$

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For any $p > 2$,there exist two symbols φ_1 and φ_2 such that $|\varphi_1^*| = |\varphi_2^*|$ $(a.e.)$, with

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"$p = \infty$" (cf lecture 2: $\alpha = 3/2$).

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"$p = \infty$" (cf lecture 2: $\alpha = 3/2$). Cannot be true for $p = 2$ (cf lecture 1) !!

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(Li–Queffélec–Rodríguez-Piazza '11-14)

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- Si φ is the lens map (of index $\theta \in (0,1)$), then

$$e^{-\alpha_\theta\sqrt{n}} \lesssim a_n(C_\varphi) \lesssim e^{-\beta_\theta\sqrt{n}}$$

q-summing operators

Suppose $1 \leq q < +\infty$ and let $T\colon X \to Y$ be a (bounded) operator between Banach spaces.

We say T is a q-summing operator if there exists $C > 0$ such that

$$\Big(\sum_{j=1}^{n} \|Tx_j\|^q\Big)^{1/q} \leq C \sup_{x^* \in B_{X^*}} \Big(\sum_{j=1}^{n} |\langle x^*, x_j\rangle|^q\Big)^{1/q} =$$

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- This forms an operator ideal.
- 1-summing operators are also called absolutely summing operators.

a (generic) example

Let (K,ν) a probability space, where K is compact and consider

$$T: \left| \begin{array}{ccc} C(K) & \longrightarrow & L^q(K,\nu) \\ f & \longmapsto & f \end{array} \right.$$

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Actually, up to factorizations, any q-summing looks like this:

Pietsch Theorem

(Pietsch '67)

$T: X \to Y$ is a q-summing operator

if and only if

there exists a (probability) measure ν on the compact (B_{X^*}, w^*) s.t.

$$\forall x \in X\ , \qquad \|T(x)\| \lesssim \Big(\int_{B_{X^*}} |\xi(x)|^q \, d\nu(\xi)\Big)^{1/q}$$

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we have the following factorization

$$\begin{array}{ccccc} X & & \xrightarrow{T} & Y & \\ \Big\downarrow & & & \Big\uparrow & \widetilde{T} \\ \widetilde{X} \subset C\big(B_{X^*}\big) & & \xrightarrow{\text{“}id\text{''}} & X_q & \subset L^q\big(B_{X^*}, \nu\big) \end{array}$$

for some probability measure ν on B_{X^*}.

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  $$j_\mu \text{ is } q_1\text{-summing} \quad \textit{if and only if} \quad j_\mu \text{ is } q_2\text{-summing}$$

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  - For $1 \leq p \leq 2$ and $q_1,\ q_2 \geq 1$.

Pietsch Theorem

Consequences

- If $q_1 \leq q_2$, every q_1-summing operator is q_2-summing.
- q-summing operators are weakly compact and map weakly convergent sequences to norm convergence sequences (Dunford-Pettis operator). In particular, they are compact when X is reflexive.
- For $1 \leq p \leq 2$, H^p and L^p have cotype 2. For $p > 2$, they have cotype p. Hence

  $$j_\mu \text{ is } q_1\text{-summing} \quad \textit{if and only if} \quad j_\mu \text{ is } q_2\text{-summing}$$

  in the following cases:

  - For $1 \leq p \leq 2$ and $q_1,\ q_2 \geq 1$.
  - For $p > 2$, and $1 \leq q_1,\ q_2 < p'$, where p' is the conjugate exponent of p.

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(Grothendieck '56)

Every operator from ℓ^1 to ℓ^2 is absolutely summing.

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Carleson embeddings

Our problem now:

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Hence we are interested in the following more general problem:

Assume from now on that μ is concentrated in the open disk $\mathbb{D}$.

For μ a Carleson measure, when the Carleson embedding

$$j_\mu \colon H^p \hookrightarrow L^p(\mu)$$

is a q-summing operator ?

Known facts

(Shapiro-Taylor '73)

Let $p \geq 2$. The composition operator $C_\varphi \colon H^p \to H^p$ is p-summing *if and only if*

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(Domenig '99)

Let $p \in [1, 2)$. There exist p-summing composition operators on H^p which are not order bounded.

First results: the annulus case.

Let us fix a finite measure μ on $\mathbb{D}$ and an integer n.

We denote by μ_n the restriction of μ to the annulus

$$\Gamma_n = \left\{z \in \mathbb{D} : 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}\right\}$$

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We have, the decomposition

$$\{f \in H^p : f(0) = 0\} = \bigoplus_{n \geq 0} H_n^p$$

which is an orthogonal decomposition when $p = 2$ (i.e. for H^2).

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Moreover

$$H^p_n \sim \ell^p_{2^n}$$

the annulus case

Let α_n be the restriction of j_n to H^p_n.

Proposition

For $1 < p < +\infty$, the following quantities are equivalent:

1. $\pi_q\big(j_n \colon H^p \to L^p(\mu_n)\big)$

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3. $\pi_q(D_a)$, where $D_a\colon \ell^p_{2^n} \to \ell^p_{2^n}$ is the diagonal operator whose multipliers are $a_j = \big(2^n\mu(R_{n,j})\big)^{1/p}$ (where $j = 1, 2, \dots, 2^n$).

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But the summing norms of multipliers on sequence spaces are known, so:

1. $1 < p \leq 2$: $\quad \pi_q(j_n) \approx \left(\sum_{j=1}^{2^n} \left[2^n \mu(R_{n,j}) \right]^{2/p} \right)^{1/2}.$

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How to glue the pieces ?

First results

In some cases, we can glue:

Theorem

In the case $q \geq p \geq 2$ we have:

$$\pi_q(j_\mu) \approx \Big(\sum_n [\pi_q(j_n)]^p\Big)^{1/p} \approx \Big(\sum_{n,j} [2^n \mu(R_{n,j})]\Big)^{1/p} \approx \Big(\int_{\mathbb{D}} \frac{1}{1-|z|}\, d\mu(z)\Big)^{1/p}.$$

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For $p > 2$, the case $1 \leq q < 2$ is still open (our tube of glue is empty...).

The case $p \leq 2$.

Thanks to the Littlewood-Paley decomposition (where $f_n \in H^p_n$)

$$\Big\| \sum_n f_n \Big\|_{H^p} \approx \Big\| \Big(\sum_n |f_n^*|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{T})}$$

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This can be used to prove

$$\Big(\sum_n \pi_2(j_n)^2 \Big)^{1/2} \lesssim \pi_2(j_\mu) \lesssim \Big(\sum_n \pi_2(j_n)^p \Big)^{1/p}$$

But none of these two estimates is the correct one.

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Our characterization is of different nature...

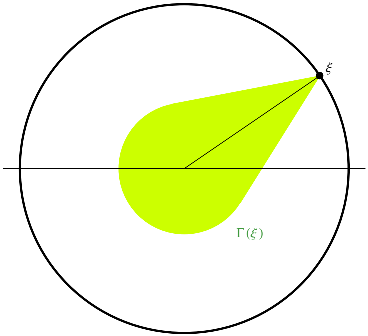
The case $p \leq 2$.

Theorem

Let $1 < p \leq 2$. The Carleson embedding $j_\mu : H^p \to L^p(\mu)$ is absolutely summing *if and only if*

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} \frac{d\mu(z)}{(1-|z|)^{1+p/2}} \right)^{2/p} d\lambda(\xi) < +\infty$$

where $\Gamma(\xi)$ is the Stolz domain in ξ:



The case $p \leq 2$: sketch of proof

Step 1 (via Maurey factorization theorem)

Let $r > 1$ with $1/r + 1/2 = 1/p$ and $T\colon X \to L^p(\mu)$ a bounded operator.

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There exists $F \in L^r(\mu)$, with $F > 0$ μ-a.e., such that $T\colon X \to L^2(\nu)$ is well defined and 2-summing, where ν is the measure defined by

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Moreover, we have

$$\pi_2\big(T\colon X \to L^p(\mu)\big)$$

$$\approx$$

$$\inf\Big\{\pi_2\big(T\colon X \to L^2(\nu)\big)\,:\, d\nu = d\mu/F^2, F \geq 0, \int F^r\, d\mu \leq 1\Big\}.$$

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Step 2

The natural injection $j\colon H^p \to L^2(\nu)$ is a 2-summing operator

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In fact we have

$$\pi_2\big(j\colon H^p \to L^2(\nu)\big) \approx \bigg(\int_{\mathbb{T}} \Big(\int_{\mathbb{D}} \frac{d\nu(z)}{|z-w|^2} \Big)^{p'/2} d\lambda(w) \bigg)^{1/p'} .$$

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Step 3

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By Ky Fan's lemma the order of taking the sup and the inf can be interchanged.

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Applying a result of Luecking, Blasco-Jarchow, we get the conclusion.

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There are many other questions of course...

- J. Shapiro: "Composition operators". Springer 1993.

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- C. Cowen, B. McCluer: "Composition operators of analytic functions" CRC Press 1995.

Merci !