Multilinear interpolation theorems with applications

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Important bilinear operators

A bounded measurable function σ on $\mathbb{R}^n \times \mathbb{R}^n$ (called a multiplier) leads to a bilinear operator W_σ defined by

$$W_\sigma(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i\langle x,\xi+\eta\rangle}\,d\xi d\eta$$

for every f, g in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

- The study of such bilinear multiplier operators was initiated by Coifman and Meyer (1978). They proved that if $1 < p, q < \infty$, $1/r = 1/p + 1/q$ and σ satisfies

  $$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \leqslant C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}$$

  for sufficiently large multi-indices α and β, then W_σ extends to a bilinear operator from $L_p(\mathbb{R}^n) \times L_q(\mathbb{R}^n)$ into $L_{r,\infty}(\mathbb{R}^n)$ whenever $r > 1$. Here as usual $L_{r,\infty}(\mathbb{R}^n)$ denotes the weak L_r space of Marcinkiewicz.

- This result was later extended to the range $1 > r > 1/2$ by Grafakos and Torres (1996) and Kenig and Stein (1999).
- Multipliers that satisfy the Marcinkiewicz condition were studied by Grafakos and Kalton (2001).
- The first significant boundedness results concerning non-smooth symbols were proved by Lacey and Thiele (1997, 1999) who established that W_σ with $\sigma(\xi,\eta) = \mathrm{sign}(\xi + \alpha\eta)$, $\alpha \in \mathbb{R} \setminus \{0,1\}$ has a bounded extension from $L_p(\mathbb{R}^n) \times L_q(\mathbb{R}^n)$ to $L_r(\mathbb{R}^n)$ if $2/3 < r < \infty$, $1 < p, q \leqslant \infty$, and $1/r = 1/p + 1/q$. Extensions of this result was subsequently obtained by Gilbert and Nahmod (2001).
- The bilinear Hilbert transform H_θ is defined for a parameter $\theta \in \mathbb{R}$ by

$$H_\theta(f,g)(x) := \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} f(x-t)g(x+\theta t)\frac{1}{t}\,dt, \quad x \in \mathbb{R}$$

  for functions f, g from the Schwartz class $\mathcal{S}(\mathbb{R})$. The family $\{H_\theta\}$ was introduced by Calderón in his study of the first commutator, an operator arising in a series decomposition of the Cauchy integral along Lipschitz curves. In 1977 Calderón posed the question whether H_θ satisfies any L_p estimates.

- In their fundamental work (Ann. of Math. **149** (1999), 475-496) Lacey and Thiele proved that if $\theta \neq -1$, then the bilinear Hilbert transform H_θ extends to a bilinear operator from $L_p \times L_q$ into L_r whenever $1 < p, q \leqslant \infty$ and $1/p + 1/q = 1/r < 3/2$.
- The bilinear Hilbert transforms arise in a variety of other related known problems in bilinear Fourier analysis, e.g., in the study of the convergence of the mixed Fourier series of the form

$$\lim_{N\to\infty} \sum_{\substack{|m-\theta n|\leqslant N \\ |m-n|\leqslant N}} \sum \hat{f}(m)\hat{g}(n)e^{2\pi i(m+n)x}.$$

- Fan and Sato in 2001 were able to show the boundedness of the bilinear Hilbert transform H on the torus $\mathbb{T}$

$$H(f,g)(x) := \int_{\mathbb{T}} f(x-t)g(x+t)\,\mathrm{ctg}(\pi t)\,dt, \quad x \in \mathbb{T}$$

by transferring the result from $\mathbb{R}$. Their proof relies upon some DeLeeuw (1969) type transference methods for multilinear multipliers.

Definition

A quasi-Banach lattice X is said to be *p-convex* if there exists a constant $C > 0$ such that for any $x_1, ..., x_n \in X$, we have

$$\Big\| \Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p} \Big\|_X \leqslant C \Big(\sum_{k=1}^{n} \|x_k\|_X^p \Big)^{1/p}.$$

The least C is denoted by $M^{(p)}(X)$.

Definition

Let X be a Banach space and $1 \leqslant p < \infty$. The sequence $\{x_n\}_{n\in\mathbb{Z}} \subset X$ is said to be *weakly p-summable* if the scalar sequences $\{x^*(x_n)\} \in \ell_p(\mathbb{Z})$ for every $x^* \in X^*$. The space of all weakly p-summing sequences in X is denoted by $\ell_p^w(X)$. It is a Banach space equipped with the norm

$$\|\{x_n\}\|_{\ell_p^w(X)} := \sup \Big\{ \Big(\sum_{n\in\mathbb{Z}} |x^*(x_n)|^p \Big)^{1/p};\ \|x^*\|_{X^*} \leqslant 1 \Big\}.$$

Bilinear interpolation theorems (M. M., 2013)

Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ and $\overline{Z} = (Z_0, Z_1)$ be quasi-Banach couples.

Definition

(i) We will say that $T := (T_0, T_1)$ is a bilinear operator from $\overline{X} \times \overline{Y}$ into $\overline{Z}$, and write $T \colon \overline{X} \times \overline{Y} \to \overline{Z}$ if $T_0 \colon X_0 \times Y_0 \to Z_0$ and $T_1 \colon X_1 \times Y_1 \to Z_1$ are bilinear operator such that $T_0(x, y) = T_1(x, y)$ for every $x \in X_0 \cap X_1$, $y \in Y_0 \cap Y_1$.

(ii) If additionally X, Y and Z are intermediate quasi-Banach spaces with respect to $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, respectively, then we say that $T \colon \overline{X} \times \overline{Y} \to \overline{Z}$ extends to a bilinear operator from $X \times Y$ into Z provided that T_0 has a bilinear extension from $X \times Y$ into Z.

Lemma

Let Y be a maximal p-convex quasi-Banach lattice on (Ω, μ) and $T\colon c_0 \times c_0 \to Y$ be a bilinear operator. If a sequence $\{a_{km}\}_{k,m\in\mathbb{Z}}$ is such that $C := \sup_{m\in\mathbb{Z}} \big(\sum_{k\in\mathbb{Z}} |a_{km}|^2\big)^{1/2} < \infty$, then the series $\sum_{k\in\mathbb{Z}} a_{km} T(e_k, e_{m-k})$ converges in Y for each $m \in \mathbb{Z}$. If we put

$$y_m := \sum_{k\in\mathbb{Z}} a_{km} T(e_k, e_{m-k}), \quad m \in \mathbb{Z}$$

then for any sequence $\{I_n\}_{n\in\mathbb{Z}}$ of disjoint sets $I_n \subset \mathbb{Z}$ and any sequence $\{\delta_m\}_{m\in\mathbb{Z}}$ of real numbers with $|\delta_m| \leqslant 1$ for each $m \in \mathbb{Z}$, we have $\big(\sum_{n\in\mathbb{Z}} \big|\sum_{m\in I_n} \delta_m y_m\big|^2\big)^{1/2} \in Y$ and

$$\Big\| \Big(\sum_{n\in\mathbb{Z}} \Big| \sum_{m\in I_n} \delta_m y_m \Big|^2 \Big)^{1/2} \Big\|_Y \leqslant C A_p^{-2} M^{(p)}(Y) \, \|T\|_{c_0\times c_0 \to Y}.$$

Definition

$\mathcal{P}$ denotes the set of all positive quasi-concave functions ρ on $(0,\infty)$, i.e. such that both ρ and $t \mapsto t\rho(1/t)$ are nondecreasing functions. We let $\mathcal{P}_0$ denote the subset of $\mathcal{P}$ consisting of all ρ such that $\rho(t) \to 0$ as $t \to 0+$, and $\rho(t)/t \to \infty$ as $t \to \infty$. On $\mathcal{P}$, we define an involution by $\rho^*(t) = 1/\rho(1/t)$ for every $t > 0$ and we put $\mathcal{P}^* := \mathcal{P}_0 \cap (\mathcal{P}_0)^*$.

Lemma

Let $\rho_0, \rho_1, \rho \in \mathcal{P}_0$ be such that $\{\rho(2^n)\} \in \ell_1 + \ell_1(2^{-n})$ and $\rho(st) \geqslant C\rho_0(s)\rho_1(t)$ for some $C > 0$ and for every $s, t > 0$. Assume that (Y_0, Y_1) is a Banach couple and

$$T\colon (c_0, c_0(2^{-n})) \times (c_0, c_0(2^{-n})) \to (Y_0, Y_1)$$

is a bilinear operator. Then we have

$$\sum_{m\in\mathbb{Z}} \Big\| \sum_{k\in\mathbb{Z}} \xi_k \eta_{m-k} T_0(e_k, e_{m-k}) \Big\|_{Y_0+Y_1} < \infty$$

for every sequences $\{\xi_n\}$ and $\{\eta_n\}$ in $c_0 \cap c_0(2^{-n})$.

Definition

Let (X_0, X_1) be a Banach couple and let $\rho \in \mathcal{P}_0$.

(i) If $\{\rho(2^n)\}_{n\in\mathbb{Z}} \in \ell_2 + \ell_2(2^{-n})$, then the space $G_{\rho,2}(X_0, X_1)$ consists of all elements $x \in X_0 + X_1$ for which $x = \sum_{n\in\mathbb{Z}} x_n$ (convergence in $X_0 + X_1$), where the elements $x_n \in X_0 \cap X_1$ are such that $\{2^{jn} x_n/\rho(2^n)\} \in \ell_2^w(X_j)$ for $j = 0, 1$. $G_{\rho,2}(X_0, X_1)$ is a Banach space equipped with the norm

$$\|x\| = \inf \max_{j=0,1} \left\| \{2^{jn} x_n/\rho(2^n)\} \right\|_{\ell_2^w(X_j)},$$

where the infimum is taken over all representations of $x = \sum_{n\in\mathbb{Z}} x_n$ as above.

(ii) The space $\langle X_0, X_1\rangle_\rho$ consists of all elements $x \in X_0 + X_1$ such that $x = \sum_{n\in\mathbb{Z}} x_n$ (convergence in $X_0 + X_1$), where the elements $x_n \in X_0 \cap X_1$ are such that $\sum_{n\in\mathbb{Z}} x_n/\rho(2^n)$ is unconditionally convergent in X_0 and $\sum_{n\in\mathbb{Z}} 2^n x_n/\rho(2^n)$ is unconditionally convergent in X_1. $\langle X_0, X_1\rangle_\rho$ is equipped with the norm

$$\|x\| = \inf \max_{j=0,1} \sup \left\| \sum_{n\in\mathbb{Z}} \lambda_n 2^{jn} x_n/\rho(2^n) \right\|_{X_j},$$

where the supremum is taken over all complex valued sequences $\{\lambda_n\}$ with $|\lambda_n| \leqslant 1$ for all n, and the infimum is taken over all representations of $x = \sum_{n\in\mathbb{Z}} x_n$.

Theorem

Let $\rho_0, \rho_1, \rho \in \mathcal{P}_0$ *be such that* $\{\rho(2^n)\} \in \ell_1 + \ell_1(2^{-n})$ *and* $\rho(st) \geqslant C\rho_0(s)\rho_1(t)$ *for some* $C > 0$ *and every* $s, t > 0$*. Assume that* (Y_0, Y_1) *is a Banach couple and* $T\colon (c_0, c_0(2^{-n})) \times (c_0, c_0(2^{-n})) \to (Y_0, Y_1)$ *is a bilinear operator. Then* T *extends to a bilinear operator*

$$\widetilde{T}\colon c_0(1/\rho_0(2^n)) \times c_0(1/\rho_1(2^n)) \to G_{\rho,2}(Y_0, Y_1).$$

Theorem

Let $\rho_0, \rho_1 \in \mathcal{P}^*$ *and* $\rho \in \mathcal{P}_0$ *be such that* $\{\rho(2^n)\} \in \ell_1 + \ell_1(2^{-n})$ *and* $\rho(st) \geqslant C\rho_0(s)\rho_1(t)$ *for some* $C > 0$ *and for every* $s, t > 0$*. Assume that* $T\colon (X_0, X_1) \times (Y_0, Y_1) \to (Z_0, Z_1)$ *is a bilinear operator between Banach couples. Then* T *extends to a bilinear operator*

$$\widetilde{T}\colon \langle X_0, X_1\rangle_{\rho_0} \times \langle Y_0, Y_1\rangle_{\rho_1} \to G_{\rho,2}(Z_0, Z_1).$$

Definition

Given a Banach couple $\overline{A} = (A_0, A_1)$, $0 \neq a \in A_0 + A_1$ and any Banach couple $\overline{X}$ an *interpolation orbit* of a in $\overline{X}$ is a Banach space

$$\mathrm{Orb}_{\overline{A}}(a, \overline{X}) := \left\{ Ta;\; T : \overline{A} \to \overline{X} \right\}$$

equipped with the norm

$$\|x\| = \inf \left\{ \|T\|_{\overline{A} \to \overline{X}};\; x = Ta,\; T : \overline{A} \to \overline{X} \right\}.$$

Lemma

Let ρ be a quasi-concave function such that $\xi_\rho := \{\rho(2^n)\} \in \ell_2 + \ell_2(2^{-n})$. Then for any Banach couple $\overline{X}$, we have

$$G_{\rho,2}(\overline{X}) = Orb_{(\ell_2, \ell_2(2^{-n}))}(\xi_\rho, \overline{X})$$

isometrically.

Bilinear operators between Calderón–Lozanovskii spaces

Definition

Let Φ the set of all functions $\varphi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that are positive, non-decreasing in each variable, and homogeneous of degree one (that is, $\varphi(\lambda s, \lambda t) = \lambda \varphi(s,t)$ for all $\lambda, s, t \geqslant 0$). Let $\varphi \in \Phi$ and $\overline{X} = (X_0, X_1)$ be a couple of quasi-Banach lattices on a measure space (Ω, μ). Following Lozanovskii, we define the space $\varphi(\overline{X}) = \varphi(X_0, X_1)$ of all $x \in L_0(\mu)$ such that $|x| = \varphi(|x_0|, |x_1|)$ for some $x_j \in X_j$, $j = 0, 1$. It is a quasi-Banach lattice equipped with the quasi-norm

$$\|x\| = \inf\Big\{\max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\};\ |x| = \varphi(|x_0|, |x_1|)\ x_j \in X_j,\ j = 0, 1\Big\}.$$

Note that if φ is concave and (X_0, X_1) is a Banach couple, then $\varphi(\overline{X})$ is a Banach lattice.

Definition

A quasi-concave function ρ is called a *quasi-power* ($\rho \in \mathcal{P}^{+-}$) if $s_\rho(t) = o(\max(1,t))$ as $t \to 0$ and $t \to \infty$, where $s_\rho(t) := \sup_{u>0} \big(\rho(tu)/\rho(u)\big)$ for every $t > 0$. The set of all $\varphi \in \Phi$ such that $\rho := \varphi(1, \cdot) \in \mathcal{P}_0$ (resp., $\rho \in \mathcal{P}^{+-}$, $\rho \in \mathcal{P}^*$) is denoted by Φ_0 (resp., Φ^{+-}, Φ^*); $\varphi_1 \sim \varphi_2$ ($\varphi_1 \overset{0}{\sim} \varphi_2$ or $\varphi_1 \overset{\infty}{\sim} \varphi_2$) means that there exist C_1, C_2 and t_0 or t_∞ such that $C_1\varphi_1(t) \leqslant \varphi_2(t) \leqslant C_2 \varphi_1(t)$ for all $t > 0$ ($0 < t \leqslant t_0$ or $t \geqslant t_\infty$, respectively).

Theorem

Let $\varphi \in \Phi^{+-}$ *be a concave function and let* $\rho(t) = \varphi(1,t)$ *for* $t > 0$*. Then for any couple* $(L_{p_0}(w_0), L_{p_1}(w_1))$ *of weighted* L_p*-spaces on a measure space, we have*

$$G_{\rho,2}(L_{p_0}(w_0), L_{p_1}(w_1)) = \varphi(L_{r_0}(w_0), L_{r_1}(w_1)),$$

where $1/r_j = \max\{0, 1/p_j - 1/2\}$ *for* $j = 0, 1$*.*

Theorem

Let $\varphi \in \Phi^{+-}$ *be a concave function such that* $\varphi(1,s)\varphi(1,t) \leqslant C\varphi(1,st)$ *for some* $C > 0$ *and every* $s, t > 0$*. If* $T\colon (X_0, X_1) \times (Y_0, Y_1) \to (L_{p_0}(w_0), L_{p_1}(w_1))$ *is a bilinear operator, then* T *extends to a bilinear operator*

$$\widetilde{T}\colon \varphi(X_0, X_1)^\circ \times \varphi(Y_0, Y_1)^\circ \to \varphi(L_{r_0}(w_0), L_{r_1}(w_1)),$$

where $1/r_j = \max\{0, 1/p_j - 1/2\}$ *for* $j = 0, 1$*.*

Bilinear operators between Calderón–Lozanovskii spaces

Corollary

Let $\rho_0, \rho_1, \rho \in \mathcal{P}_0$ *be such that* $\{\rho(2^n)\} \in \ell_1 + \ell_1(2^{-n})$ *and* $\rho(st) \geqslant C\rho_0(s)\rho_1(t)$ *for some* $C > 0$ *and every* $s, t > 0$*. If* $T\colon (c_0, c_0(2^{-n})) \times (c_0, c_0(2^{-n})) \to (\ell_1, \ell_1(2^n))$*, then* T *extends to a bilinear operator*

$$\widetilde{T}\colon c_0(1/\rho_0(2^n)) \times c_0(1/\rho_1(2^n)) \to \ell_2(1/\rho(2^{-n})).$$

Bilinear operators between Calderón–Lozanovskii spaces

Definition

A quasi-Banach space X is said to be p-normable if there exists a constant C such that for any $x_1, \ldots, x_n \in X$ we have

$$\Big\| \sum_{j=1}^{n} x_j \Big\|_X \leqslant C \Big(\sum_{j=1}^{n} \|x_j\|_X^p \Big)^{1/p}.$$

It is well known that a fundamental theorem of Aoki and Rolewicz asserts that every quasi-Banach space is p-normable for some $0 < p \leqslant 1$.

Theorem

Let $\varphi_0, \varphi_1, \varphi \in \mathcal{P}^*$ and $\varphi \in \Phi^*$ be such that $\{\varphi(1, 2^n)\} \in \ell_p + \ell_p(2^{-n})$ for some $0 < p \leqslant 1$ and let $\sup_{m\in\mathbb{Z}} \big(\sum_{k\in\mathbb{Z}} \big(\rho_0(2^k)\rho_1(2^{k-m})/\varphi(1, 2^m)\big)^2\big)^{1/2} < \infty$. Assume that $T\colon (X_0, X_1) \times (Y_0, Y_1) \to (Z_0, Z_1)$ is a bilinear operator between couples of quasi-Banach lattices. If both Z_0 and Z_1 are p-normable with nontrivial concavity, then T extends to a bilinear operator

$$\widetilde{T}\colon \varphi_0(X_0, X_1)^\circ \times \varphi_1(Y_0, Y_1)^\circ \to \varphi(Z_0, Z_1).$$

Definition

We define a subclass $\mathcal{P}_p$ ($1 \leqslant p < \infty$) of $\mathcal{P}$ containing of all ρ satisfying the condition:

$$\sup_{m\in\mathbb{Z}} \frac{1}{\rho(2^m)} \Big(\sum_{k\in\mathbb{Z}} (\rho(2^k)\rho(2^{m-k}))^2 \Big)^{1/2} < \infty.$$

Theorem

Let $\varphi \in \Phi^{+-}$ be such that $\rho = \varphi(1,\cdot) \in \mathcal{P}_2$. Assume that $T\colon (X_0, X_1) \times (Y_0, Y_1) \to (Z_0, Z_1)$ is a bilinear operator between quasi-Banach lattices. If Z_0 and Z_1 have nontrivial concavity, then T extends to a bilinear operator

$$\widetilde{T}\colon \varphi(X_0, X_1)^\circ \times \varphi(Y_0, Y_1)^\circ \to \varphi(Z_0, Z_1).$$

Theorem

Let $\varphi \in \Phi^{+-}$ be such that $\rho = \varphi(1,\cdot) \in \mathcal{P}_2$. Assume that $T\colon (X_0, X_1) \times (Y_0, Y_1) \to (Z_0, Z_1)$ is a bilinear operator between Banach couples. Then T extends to a bilinear operator

$$\widetilde{T}\colon \langle X_0, X_1\rangle_\rho \times \langle Y_0, Y_1\rangle_\rho \to \varphi_u(Z_0, Z_1).$$

We give examples of functions satisfying a stronger condition than the one required in proceeding theorems. Following Astashkin, we define ϕ by

$$\phi(t) = \begin{cases} t^a \ln^c(C_1/t), & 0 < t \leqslant 1, \\ t^b \ln^d(C_2\, t), & t > 1, \end{cases}$$

where $0 < a < b < 1$, $c > 1$, $d > 1$ and constants $C_1 > e^{c/a}$, $C_2 > e^d d/(1-b)$ are chosen such that ϕ is continuous. Then ϕ is a quasi-power and satisfies

$$\sup_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \frac{\phi(2^m)}{\phi(2^k)\phi(2^{m-k})} < \infty.$$

In consequence ρ defined by $\rho(t) := t/\phi(t)$ for every $t > 0$ satisfies

$$\sup_{m\in\mathbb{Z}} \frac{1}{\rho(2^m)} \sum_{k\in\mathbb{Z}} \rho(2^k)\rho(2^{m-k}) < \infty,$$

and whence $\rho \in \mathcal{P}_2$.

Lemma

Suppose $\rho_0, \rho_1 \in \mathcal{P}_p$ for some $1 \leqslant p < \infty$. If $\varphi \in \Phi$ is such that $C\varphi(1, st) \geqslant \varphi(1, s)\varphi(1, t)$ for some $C > 0$ and every $s, t > 0$, then $\rho \in \mathcal{P}_p$, where $\rho(t) := \varphi(\rho_0(t), \rho_1(t))$ for all $t > 0$.

Applications

We recall that if ψ is an Orlicz function (i.e., $\psi\colon [0,\infty) \to [0,\infty)$ is increasing, continuous and $\psi(0) = 0$), then the Orlicz space L_ψ on a given measure space (Ω, μ) is defined to be a subspace of $L_0(\mu)$ consisting of all $f \in L_0(\mu)$ such that for some $\lambda > 0$ holds

$$\int_\Omega \psi(\lambda |f|)\, d\mu < \infty.$$

If there exists $C > 0$ such that $\psi(t/C) \leqslant \psi(t)/2$ for every $t > 0$, then L_ψ is a quasi-Banach lattice with the quasi-norm $\|\cdot\|$ satisfying

$$\|f + g\| \leqslant C(\|f\| + \|g\|), \quad f, g \in L_\psi,$$

where $\|f\| := \inf\left\{\lambda > 0;\ \int_\Omega \psi\left(\frac{|f|}{\lambda}\right) d\mu \leqslant 1\right\}$. A simple calculation shows for any couple (L_{p_0}, L_{p_1}) of L_p-spaces on a measure space (Ω, μ) with $0 < p_0, p_1 \leqslant \infty$, we have

$$\varphi(L_{p_0}, L_{p_1}) = L_\psi$$

with equivalence of the quasi-norms, where $\psi^{-1}(t) = \varphi(t^{1/p_0}, t^{1/p_1})$ for every $t \geqslant 0$.

Theorem

Assume $\rho \in \mathcal{P}_2$, $1 \leqslant u_j, v_j < \infty$ *and* $0 < s_j < \infty$ *for* $j = 0, 1$ *and let* $T\colon (L_{u_0}, L_{u_1}) \times (L_{v_0}, L_{v_1}) \to (L_{s_0}, L_{s_1})$ *be a bilinear operator between couples of* L_p *spaces. Then* T *extends to a bounded bilinear operator*

$$\widetilde{T}\colon L_{\psi_0} \times L_{\psi_1} \to L_{\psi}$$

between Orlicz spaces with

- $\psi_0^{-1}(t) \sim t^{1/u_0}\rho(t^{1/u_1 - 1/u_0})$,
- $\psi_1^{-1}(t) \sim t^{1/v_0}\rho(t^{1/v_1 - 1/v_0})$,
- $\psi^{-1}(t) \sim t^{1/s_0}\rho(t^{1/s_1 - 1/s_0})$.

An important class of Orlicz spaces are Zygmund classes.

For $0 < \alpha, \beta < \infty$ and $0 < p, q < \infty$ the Zygmund space $\mathcal{Z}_{p,q}^{\alpha,\beta}$ on a measure space (Ω, μ) consists of all $f \in L_0(\mu)$ such that

$$\int_{\{|f|\leqslant 1\}} |f|^p (1 - \log|f|)^\alpha \, d\mu + \int_{\{|f|>1\}} |f|^q (1 + \log|f|)^\beta \, d\mu < \infty.$$

It is clear that if $\psi\colon [0,\infty) \to [0,\infty)$ is an Orlicz function such that $\psi(t) \overset{0}{\sim} t^p(1-\log t)^\alpha$ and $\psi(t) \overset{\infty}{\approx} t^q(1+\log t)^q$, then $\mathcal{Z}_{p,q}^{\alpha,\beta}$ coincides with L_Φ. We equipped $\mathcal{Z}_{p,q}^{\alpha,\beta}$ with the quasi-norm $\|\cdot\|_\Phi$ generated by shown Φ.

Theorem

For every $\theta \neq -1$ the bilinear Hilbert transform H_θ extends to a bilinear operator from $\mathcal{Z}_{p_0,q_0}^{\alpha_0,\beta_0} \times \mathcal{Z}_{p_1,q_1}^{\alpha_1,\beta_1}$ into $\mathcal{Z}_{p,q}^{\alpha,\beta}$ provided $1 < p_0 < q_0 < \infty$, $1 < p_1 < q_1 < \infty$, $1/p_0 + 1/p_1 = 1/p < 3/2$, $\alpha_0/p_0 = \alpha_1/p_1 = \alpha/p > 1$ and $\beta_0/q_0 = \beta_1/q_1 = \beta/q > 1$.

Interpolation of analytic families of multilinear operators (L. Grafakos & M. M., 2014)

- Stein's interpolation theorem [Trans. Amer. Math. Soc. (1956)] for analytic families of operators between L^p spaces $(p \geqslant 1)$ has found several significant applications in harmonic analysis. This theorem provides a generalization of the classical single-operator Riesz-Thorin interpolation theorem to a family $\{T_z\}$ of operators that depend analytically on a complex variable z.
- In the framework of Banach spaces, interpolation for analytic families of multilinear operators can be obtained via duality in a way similar to that used in the linear case. For instance, one may adapt the proofs in Zygmund book and Berg and Löfstrom for a single multilinear operator to a family of multilinear operators. However, this duality-based approach is not applicable to quasi-Banach spaces since their topological dual spaces may be trivial.

The open strip $\{z;\, 0 < \mathrm{Re}\, z < 1\}$ in the complex plane is denoted by S, its closure by $\overline{S}$ and its boundary by ∂S.

Definition Let $A(S)$ be the space of scalar-valued functions, analytic in S and continuous and bounded in $\overline{S}$. For a given couple (A_0, A_1) of quasi-Banach spaces and A another quasi-Banach space satisfying $A \subset A_0 \cap A_1$, we denote by $\mathcal{F}(A)$ the space of all functions $f\colon S \to A$ that can be written as finite sums of the form

$$f(z) = \sum_{k=1}^{N} \varphi_k(z) a_k, \quad z \in \overline{S},$$

where $a_k \in A$ and $\varphi_k \in A(S)$. For every $f \in \mathcal{F}(A)$ we set

$$\|f\|_{\mathcal{F}(A)} = \max\Big\{ \sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1} \Big\}.$$

Remark Clearly we have that $\|a\|_\theta \leqslant \|a\|_{A_0\cap A_1}$ for every $a \in A_0 \cap A_1$, and notice that $\|\cdot\|_\theta$ could be identically zero.

Definition A quasi-Banach couple is said to be admissible whenever $\|\cdot\|_\theta$ is a quasi-norm on $A_0 \cap A_1$, and in this case, the quasi-normed space $(A_0 \cap A_1, \|\cdot\|_\theta)$ is denoted by $(A_0, A_1)_\theta$.

Remark If A is dense in $A_0 \cap A_1$, then for every $a \in A$ we have

$$\|a\|_\theta = \inf\{\|f\|_{\mathcal{F}(A)};\ f \in \mathcal{F}(A),\ f(\theta) = a\}.$$

Definition If there is a completion of $(A_0, A_1)_\theta$ which is set-theoretically contained in $A_0 + A_1$, then it is denoted by $[A_0, A_1]_\theta$.

Definition A continuous function $F\colon \overline{S} \to \mathbb{C}$ which is analytic in S is said to be of admissible growth if there is $0 \leqslant \alpha < \pi$ such that

$$\sup_{z\in\overline{S}} \frac{\log|F(z)|}{e^{\alpha|\mathrm{Im}\, z|}} < \infty.$$

Lemma [I. I. Hirchman, J. Analyse Math. (1953)] If a function $F\colon \overline{S} \to \mathbb{C}$ is analytic, continuous on $\overline{S}$, and is of admissible growth, then

$$\log|F(\theta)| \leqslant \int_{-\infty}^{\infty} \log|F(it)|\, P_0(\theta, t)\, dt + \int_{-\infty}^{\infty} \log|F(1+it)|\, P_1(\theta, t)\, dt,$$

where P_j $(j = 0, 1)$ are the Poisson kernels for the strip given by

$$P_j(x+iy, t) = \frac{e^{-\pi(t-y)}\sin \pi x}{\sin^2 \pi x + (\cos \pi x - (-1)^j e^{-\pi(t-y)})^2}, \quad x + iy \in \overline{S}.$$

Definition Let (Ω, Σ, μ) be a measure space and let $\mathcal{X}_1,\dots,\mathcal{X}_m$ be linear spaces. The family $\{T_z\}_{z\in\overline{S}}$ of multilinear operators $T\colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to \widetilde{L}^0(\mu)$ is said to be analytic if for any $(x_1,\dots,x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ and for almost every $\omega \in \Omega$ the function

$$z \mapsto T_z(x_1,\dots,x_m)(\omega), \quad z \in \overline{S}$$

is analytic in S and continuous on $\overline{S}$. Additionally, if for $j = 0$ and $j = 1$ the function

$$(t,\omega) \mapsto T_{j+it}(x_1,\dots,x_n)(\omega), \quad (t,\omega) \in \mathbb{R} \times \Omega \qquad (*)$$

is $(\mathcal{L} \times \Sigma)$-measurable for every $(x_1,\dots,x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, and for almost every $\omega \in \Omega$ the function given by formula $(*)$ is of admissible growth, then the family $\{T_z\}$ is said to be an admissible analytic family. Here $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$.

Theorem For each $1 \leqslant i \leqslant m$, let $\overline{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j-convex for $j = 0, 1$. Assume that $\mathcal{X}_i$ is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \leqslant i \leqslant m$, and that $\{T_z\}_{z \in \overline{S}}$ is an admissible analytic family of multilinear operators $T_z \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to Y_0 \cap Y_1$. Suppose that for every $(x_1, \ldots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$,

$$\|T_{j+it}(x_1, \ldots, x_m)\|_{Y_j} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot)\, dt)$ for all $\theta \in (0,1)$. Then for all $(x_1, \ldots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, all $s \in \mathbb{R}$, and all $0 < \theta < 1$ we have

$$\|T_{\theta+is}(x_1, \ldots, x_m)\|_{Y_0^{1-\theta} Y_1^{\theta}} \leqslant \big(M^{(p_0)}(Y_0)\big)^{1-\theta} \big(M^{(p_1)}(Y_1)\big)^{\theta} K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s)\, dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s)\, dt.$$

Lemma Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space (Ω, Σ, μ) such that X_0 is p_0-convex and X_1 is p_1-convex. Then for every $0 < \theta < 1$ we have

$$\|x\|_{X_0^{1-\theta} X_1^{\theta}} \leqslant (M^{(p_0)}(X_0))^{1-\theta} (M^{(p_1)}(X_1))^{\theta} \, \|x\|_{(X_0, X_1)_\theta}, \quad x \in X_0 \cap X_1.$$

In particular (X_0, X_1) is an admissible quasi-Banach couple.

Lemma Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure (Ω, Σ, μ). If $x_j \in X_j$ are such that $|x_j|$ $(j = 0, 1)$ are bounded above and their non-zero values have positive lower bounds, then

$$|x_0|^{1-\theta} |x_1|^{\theta} \in (X_0, X_1)_\theta$$

and

$$\big\| |x_0|^{1-\theta} |x_1|^{\theta} \big\|_{(X_0, X_1)_\theta} \leqslant \|x_0\|_{X_0}^{1-\theta} \|x_1\|_{X_1}^{\theta}.$$

Corollary Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space (Ω, Σ, μ). If $x \in X_0 \cap X_1$ has an order continuous norm in $X_0^{1-\theta} X_1^{\theta}$, then for every $0 < \theta < 1$,

$$\|x\|_{(X_0,X_1)_\theta} \leqslant \|x\|_{X_0^{1-\theta} X_1^{\theta}}.$$

Theorem Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space with nontrivial lattice convexity constants. If the space $X_0^{1-\theta} X_1^{\theta}$ has order continuous quasi-norm, then

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^{\theta}$$

up equivalences of norms (isometrically, provided that lattice convexity constants are equal to 1). In particular this holds if at least one of the spaces X_0 or X_1 is order continuous.

Theorem For each $1 \leqslant i \leqslant m$, let (X_{0i}, X_{1i}) be complex quasi-Banach function lattices and let Y_j be complex p_j-convex maximal quasi-Banach function lattices with p_j-convexity constants equal 1 for $j = 0, 1$. Suppose that either X_{0i} or X_{1i} is order continuous for each $1 \leqslant i \leqslant m$. Let T be a multilinear operator defined on $(X_{01} + X_{11}) \times \cdots \times (X_{0m} + X_{1m})$ and taking values in $Y_0 + Y_1$ such that

$$T\colon X_{i1} \times \cdots \times X_{im} \to Y_i$$

is bounded with quasi-norm M_i for $i = 0, 1$. Then for $0 < \theta < 1$,

$$T\colon (X_{01})^{1-\theta}(X_{11})^{\theta} \times \cdots \times (X_{0m})^{1-\theta}(X_{1m})^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$$

is bounded with the quasi-norm

$$\|T\| \leqslant M_0^{1-\theta}\, M_1^{\theta}.$$

As an application we obtain the following interpolation theorem for operators was proved by Kalton (1990), which was applied to study a problem in uniqueness of structure in quasi-Banach lattices (Kalton's proof uses a deep theorem by Nikishin and the theory of Hardy H_p-spaces on the unit disc).

Theorem Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measures spaces. Let X_i, $i = 0, 1$, be complex p_i-convex quasi-Banach lattices on $(\Omega_1, \Sigma_1, \mu_1)$ and let Y_i be complex p_i-convex maximal quasi-Banach lattices on $(\Omega_2, \Sigma_2, \mu_2)$ with p_i-convexity constants equal 1. Suppose that either X_0 or X_1 is order continuous. Let $T\colon X_0 + X_1 \to L^0(\mu_2)$ be a continuous operator such that $T(X_0) \subset Y_0$ and $T(X_1) \subset Y_1$. Then for $0 < \theta < 1$,

$$T\colon X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$$

and

$$\|T\|_{X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}} \leqslant \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}.$$

Applications to Hardy spaces

Suppose that there is an operator $\mathcal{M}$ defined on a linear subspace of $\widetilde{L}^0(\Omega, \Sigma, \mu)$ and taking values in $\widetilde{L}^0(\Omega, \Sigma, \mu)$ such that:

(a) For $j = 0$ and $j = 1$ the function $(t, x) \mapsto \mathcal{M}(h(j + it, \cdot))(\omega)$, $(t, \omega) \in \mathbb{R} \times \Omega$ is $\mathcal{L} \times \Sigma$-measurable for any function $h\colon \partial S \times \Omega \to \mathbb{C}$ such that $\omega \mapsto h(j + it, \omega)$ is Σ-measurable for almost all $t \in \mathbb{R}$.

(b) $\mathcal{M}(\lambda h)(\omega) = |\lambda| \mathcal{M}(h)(\omega)$ for all $\lambda \in \mathbb{C}$.

(c) For every function h as in above there is an exceptional set $E_h \in \Sigma$ with $\mu(E_h) = 0$ such that for $j \in \{0, 1\}$

$$\mathcal{M}\left(\int_{-\infty}^{\infty} h(t, \cdot) P_j(\theta, t)\, dt\right)(\omega) \leqslant \int_{-\infty}^{\infty} \mathcal{M}(h(t, \cdot))(\omega) P_j(\theta, t)\, dt$$

for all $z \in \mathbb{C}$, all $\theta \in (0, 1)$, and all $\omega \notin E_h$. Moreover, $E_{\psi h} = E_h$ for every analytic function ψ on S which is bounded on $\overline{S}$.

For each $1 \leqslant i \leqslant m$, let $\overline{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of complex maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j-convex for $j = 0, 1$. Assume that $\mathcal{X}_i$ is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \leqslant i \leqslant m$, and that $\{T_z\}_{z \in \overline{S}}$ is an admissible analytic family of multilinear operators $T_z \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to Y_0 \cap Y_1$. Assume that $\mathcal{M}$ is defined on the range of T_z, takes values in $L^0(\Omega, \Sigma, \mu)$, and satisfies conditions (a), (b) and (c).

Theorem Suppose that for every $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and

$$\|\mathcal{M}(T_{j+it}(x_1, \dots, x_m))\|_{Y_j} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}, \quad j = 0, 1,$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot)\, dt)$ for all $\theta \in (0, 1)$. Then for all $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $s \in \mathbb{R}$, and $0 < \theta < 1$,

$$\|\mathcal{M}(T_{\theta+is}(x_1, \dots, x_m))\|_{Y_0^{1-\theta} Y_1^{\theta}} \leqslant C_\theta\, K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$C_\theta = \big(M^{(p_0)}(Y_0)\big)^{1-\theta} \big(M^{(p_1)}(Y_1)\big)^{\theta},$$

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s)\, dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s)\, dt.$$

The preceding theorem has an important application to interpolation of multilinear operators that take values in Hardy spaces. A particular case of the above Theorem arises when:

- $\Omega = \mathbb{R}^n$, μ is Lebesgue measure, and

$$\mathcal{M}(h)(x) = \sup_{\delta>0} |\phi_\delta * h(x)|, \quad x \in \mathbb{R}^n$$

where ϕ is a Schwartz function on $\mathbb{R}^n$ with nonvanishing integral.

- $Y_0 = L^{p_0}$, $Y_1 = L^{p_1}$, in which case $Y_0^{1-\theta} Y_1^{\theta} = L^p$, where $1/p = (1-\theta)/p_0 + \theta/p_1$.

Definition The classical Hardy space H^p of Fefferman and Stein is defined by

$$\|h\|_{H^p} := \|\mathcal{M}(h)\|_{L^p}.$$

Corollary If $\{T_z\}$ is an admissible analytic family is such that

$$\|T_{j+it}(x_1, \ldots, x_m)\|_{H^{p_j}} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}, \quad j = 0, 1,$$

then

$$\|T_{\theta+s}(x_1, \ldots, x_m)\|_{H^p} \leqslant K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta}$$

for $0 < p_0, p_1 < \infty$, $s \in \mathbb{R}$, and $0 < \theta < 1$. Analogous estimates hold for the Hardy-Lorentz spaces $H^{q,r}$ where estimates of the form

$$\|T_{j+it}(x_1, \ldots, x_m)\|_{H^{q_j, r_j}} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}$$

for admissible analytic families $\{T_z\}$ when $j = 0, 1$ imply

$$\|T_{\theta+is}(x_1, \ldots, x_m)\|_{H^{q,r}} \leqslant C\, K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$C = 2^{\frac{1}{q}} \left(\frac{u\, q_0^{1-\theta} q_1^{\theta}}{\log 2}\right)^u \left(\frac{q_0}{q_0 - p_0}\right)^{\frac{1-\theta}{p_0}} \left(\frac{q_1}{q_1 - p_1}\right)^{\frac{\theta}{p_1}},$$

$0 < p_j < q_j < \infty$, $p_j \leqslant r_j \leqslant \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$ while $u = 1$ if $1 < q_0, q_1 \leqslant \infty$ and $1 \leqslant r_0, r_1 \leqslant \infty$

An application to the bilinear Bochner-Riesz operators

Stein's motivation to study analytic families of operators might have been the study of the Bochner-Riesz operators

$$B^{\delta}(f)(x) := \int_{|\xi|\leqslant 1} \left(1 - |\xi|^2\right)^{\delta} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

in which the "smoothness" variable δ affects the degree p of integrability of $B^{\delta}(f)$ on $L^p(\mathbb{R}^n)$. Here f is a Schwartz function on $\mathbb{R}^n$ and $\widehat{f}$ is its Fourier transform defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \quad \xi \in \mathbb{R}^n.$$

Remark Using interpolation for analytic families of operators, Stein showed that whenever $\delta > (n-1)|1/p - 1/2|$, then

$$B^{\delta}\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is bounded for every $1 \leqslant p \leqslant \infty$.

- The bilinear Bochner-Riesz operators are defined on $\mathcal{S} \times \mathcal{S}$ by

$$S^{\delta}(f,g)(x) := \iint_{|\xi|^2+|\eta|^2 \leqslant 1} \left(1 - |\xi|^2 - |\eta|^2\right)^{\delta} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta$$

for every $f, g \in \mathcal{S}$.

- The bilinear Bochner-Riesz means S^z is defined by

$$S^z(f,g)(x) = \int\int K_z(x - y_1, x - y_2) f(y_1) g(y_2) dy_1\, dy_2,$$

where that the kernel of $S^{\delta+it}$ is given by

$$K_{\delta+it}(x_1,x_2) = \frac{\Gamma(\delta+1+it)}{\pi^{\delta+it}} \frac{J_{\delta+it+n}(2\pi|x|)}{|x|^{\delta+it+n}}, \quad x = (x_1,x_2).$$

If $\delta > n - 1/2$, then using known asymptotics for Bessel functions we have that this kernel satisfies an estimate of the form:

$$|K_{\delta+it}(x_1, x_2)| \leqslant \frac{C(n+\delta+it)}{(1+|x|)^{\delta+n+1/2}},$$

where $C(n+\delta+it)$ is a constant that satisfies

$$C(n+\delta+it) \leqslant C_{n+\delta} e^{B\,|t|^2}$$

for some $B > 0$ and so we have

$$|K_{\delta+it}(x_1, x_2)| \leqslant C_{n+\delta}\, e^{B|t|^2} \frac{1}{(1+|x_1|)^{n+\epsilon}} \frac{1}{(1+|x_2|)^{n+\epsilon}},$$

with $\epsilon = \frac{1}{2}(\delta - n - 1/2)$. It follows that the bilinear operator $S^{\delta+it}$ is bounded by a product of two linear operators, each of which has a good integrable kernel. It follows that

$$K^{\delta+it} \colon L^1 \times L^1 \to L^{1/2}$$

with constant $K_1(t) \leqslant C'_{n+\delta} e^{B|t|^2}$ whenever $\delta > n - 1/2$.

Theorem Let $1 < p < 2$. For any $\lambda > (2n-1)(1/p - 1/2)$

$$S^{\lambda}\colon L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \to L^{p/2}(\mathbb{R}^n) \quad \textit{is} \quad \textit{bounded}.$$