Decomposition norm theorem, L^p-behavior of reproducing kernels and two weight inequality for Bergman projection.

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Joint works with O. Constantin and J. Rättyä

VI International Course of Mathematical Analysis in Andalucía
Antequera

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- Projections. Some facts on the one weight problem.

Which are those weights (ω, v) satisfying the two weight inequality

$$\|P_\omega(f)\|_{L^p_v} \lesssim \|f\|_{L^p_v}, \quad f \in L^p_v?$$

A classification of radial weights

Regular weights.

A radial weight is called regular, $\omega \in \mathcal{R}$, if $\omega \in \widehat{\mathcal{D}}$ and

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- $\mathcal{R}$ is a natural framework for an extension of the classical theory on the standard Bergman spaces A^p_α.

Rapidly increasing weights.

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- $$H^p \subset A^p_\omega \subset \cap_{\alpha>-1} A^p_\alpha, \quad \omega \in \mathcal{I}.$$

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- Exponential type weights

$$\omega(r) = \exp\left(-\frac{C}{(1-r)^\alpha}\right), \quad C, \alpha > 0$$

are rapidly decreasing.

Theorem. (Dostanic (2004))

If $\omega(r) = (1-r^2)^A \exp\left(\frac{-B}{(1-r^2)^\alpha}\right)$, $A > 0$, $B > 0, 0 < \alpha \leq 1$. Then, the Bergman projection is bounded from L^p_ω to L^p_ω only for $p = 2$.

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Theorem. Zeytuncu (2012), Constantin-P (2014)

Assume that $\omega(r) = e^{-2\phi(r)}$ is a radial weight such that $\phi : [0,1) \to \mathbb{R}^+$ is a C^∞-function, ϕ' is positive on $[0,1)$, $\lim_{r\to 1^-} \phi(r) = \lim_{r\to 1^-} \phi'(r) = +\infty$ and

$$\lim_{r\to 1^-} \frac{\phi^{(n)}(r)}{(\phi'(r))^n} = 0, \quad \text{for any } n \in \mathbb{N} \setminus \{1\}.$$

Then, the Bergman projection is bounded from L^p_ω to L^p_ω only for $p = 2$.

Theorem. Constantin-P (2014)

Let $v(r) = \exp\left(-\frac{\alpha}{1-r}\right), \alpha > 0$, and $1 \leq p < \infty$. Then, the Bergman projection

$$P_v(f)(z) = \int_{\mathbb{D}} f(\zeta)\, B^v(z,\zeta) v(z)\, dm(\zeta)$$

is bounded from $L^p\left(\mathbb{D}, v^{p/2}\right)$ to $A^p_{v^{p/2}}$.

Proposition. Constantin-P (2014)

Let $v(r) = \exp\left(-\frac{\alpha}{1-r}\right)$, $\alpha > 0$, and let $B^v(z) = \sum_{n=0}^{\infty} \frac{z^n}{v_{2n+1}}$. Then, there is a positive constant C such that

$$M_1(r, B^v) \asymp \frac{\exp\left(\frac{\alpha}{1-\sqrt{r}}\right)}{(1-r)^{\frac{3}{2}}}, \quad r \to 1^-,$$

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- A more general result have been obtained by Arrousi-Pau (2014) by using (Marzo-Ortega approach for weighted Fock spaces) and Hörmander (Berndtsson) L^2-estimates for solutions of the $\bar{\partial}$-equation.

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- If $\omega(z) = (1-|z|^2)^\alpha$ and $v(z) = (1-|z|^2)^\beta$,

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$$\|B_a^\omega\|_{A_v^p}^p \asymp \int_0^{|a|} \frac{1}{(1-r)^{(2+\alpha)p-\beta+1}}\, dr \asymp \int_0^{|a|} \frac{\widehat{v}(r)}{\widehat{\omega}(r)^p(1-r)^p}\, dr, \quad |a| \to 1^-.$$

Theorem (P-Rättyä 2014)

Let $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and $N \in \mathbb{N} \cup \{0\}$. Then the following assertions hold:

(i) $M_p^p\left(r, (B_a^\omega)^{(N)}\right) \asymp \int_0^{|a|r} \frac{dt}{\widehat{\omega}(t)^p(1-t)^{p(N+1)}}, \quad r, |a| \to 1^-.$

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The inequality $\lesssim$ in (ii) is actually valid for any radial weight v.

Corollary (P-Rättyä 2014)

Let $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and $N \in \mathbb{N} \cup \{0\}$. Then the following assertions hold.

(i) $M_p^p\left(r, (B_a^\omega)^{(N)}\right) \asymp \dfrac{1}{\widehat{\omega}(ar)^p(1-|a|r)^{p(N+1)-1}}, \quad r, |a| \to 1^-,$

if and only if

$$\int_0^{|a|} \frac{dt}{\widehat{\omega}(t)^p(1-t)^{p(N+1)}} \lesssim \frac{1}{\widehat{\omega}(a)^p(1-|a|)^{p(N+1)-1}}, \quad |a| \to 1^-.$$

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(ii) If $v \in \widehat{\mathcal{D}}$, then

$$\| (B_a^\omega)^{(N)} \|_{A_v^p}^p \asymp \frac{\widehat{v}(a)}{\widehat{\omega}(a)^p(1-r)^{p(N+1)-1}}, \quad |a| \to 1^-,$$

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Sketch of the proof for $\lesssim$ of (ii). $v \in \mathcal{R}$. $p > 1$

$$\|B_a^\omega\|_{A_v^p}^p \asymp \sum_{n=0}^{\infty} 2^{-n} \|\Delta_n^v B_a^\omega\|_{H^p}^p$$

$$\asymp \sum_{n=0}^{\infty} 2^{-n} \left\| \sum_{k \in I_v(n)} \frac{(\bar{a}z)^n}{2\omega_{2n+1}} \right\|_{H^p}^p \quad \text{Decomposition norm}$$

$$\asymp \sum_{n=0}^{\infty} \frac{2^{-n}}{\omega^p_{2E\left(\frac{1}{1-r_n}\right)+1}} \left\| \sum_{k \in I_v(n)} (\bar{a}z)^n \right\|_{H^p}^p \quad \text{Moments of } \omega \text{ are smooth}$$

$$\le \sum_{n=0}^{\infty} \frac{2^{-n}}{\omega^p_{2E\left(\frac{1}{1-r_{n+1}}\right)+1}} |a|^{E\left(\frac{1}{1-r_n}\right)} \left\| \sum_{k \in I_v(n)} z^n \right\|_{H^p}^p$$

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$$\asymp \int_0^{|a|} \frac{\widehat{v}(t)}{\widehat{\omega}(t)^p (1-t)^p}\, dt \quad \omega \text{ and } v \text{ are doubling}$$

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3. $\frac{v(z)}{(1-|z|^2)^\alpha} \in B_p(\alpha)$. That is,

$$B_{p,\alpha}(v) = \sup_{I\subset\mathbb{T}} \frac{\left(\int_{S(I)} v(z)\, dA_\alpha(z)\right)\left(\int_{S(I)} v(z)^{\frac{-p'}{p}}\, dA_\alpha(z)\right)^{\frac{p}{p'}}}{|I|^{(2+\alpha)p}} < \infty.$$

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- This does not remain true for a general weight

Question

What is known about the two-weight inequality

$$\|P_\omega(f)\|_{L^p_v} \lesssim \|f\|_{L^p_v}, \quad f \in L^p_v$$

when ω is not an standard weight?

Case $p > 1$.

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- This is not true for the Bekollé-Bonami condition $B_p(\alpha)$!!

Sketch of the proof of $(b) \Rightarrow (c)$

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- The adjoint of P_ω, with respect to $\langle \cdot, \cdot \rangle_{L^2_v}$, is given by

$$P_\omega^\star(g)(\zeta) = \frac{\omega(\zeta)}{v(\zeta)} \int_{\mathbb{D}} g(z) B^\omega(\zeta, z) v(z)\, dA(z), \quad g \in L^{p'}_v.$$

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- $P^\star_\omega : L^{p'}_v \to L^{p'}_v$ is bounded, and hence, by choosing $g_n(z) = z^n$, $n \in \mathbb{N}$, as test functions, we get

$$\sup_{0<r<1} \frac{\widehat{v}(r)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds \right)^{\frac{1}{p'}}}{\widehat{\omega}(r)} < \infty.$$

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Lemma. (P-Rättyä (2014))

Let ω be a radial weight and $1 < p < \infty$. Denote $\omega_1(r) = \omega(r)^{1-p}(1-r)^{-p}$ and $\omega_2(r) = (\omega(r)(1-r))^{-\frac{1}{p}}\omega(r)$. Then the following assertions are equivalent:

(i) $\omega \in \mathcal{R}$;

(ii) ω satisfies the (LS)-property and

$$\sup_{0<r<1} \left(\frac{\widetilde{\psi}_{\omega_1}(r)}{1-r} \right) \left(\frac{\psi_\omega(r)}{1-r} \right)^{p-1} < \infty;$$

(iii) $\dfrac{\widetilde{\psi}_{\omega_1}(r)}{1-r} \asymp 1, \quad r \to 1^-$;

(iv) $\omega_2 \in \mathcal{R}$.

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- Some calculations. . . ...

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Let $1 < p < \infty$.

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- Part (i) follows from the two weight inequality.
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- The proof of (iii) is a little bit more involved, relies on them and on a result of Muckenhoupt on Hardy operators.

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$$\asymp K(|z|)\int_{|z|}^1 \phi(s)\omega(s)\,ds.$$

- $$\|P_\omega^+(\phi)\|_{L^p_\omega}^p \gtrsim \int_0^1 \left(K(r)\int_r^1 \phi(s)\omega(s)\,ds\right)^p \omega(r)\,dr,$$
- $$\int_0^1 \left(K(r)\int_r^1 \phi(s)\omega(s)\,ds\right)^p \omega(r)\,dr \lesssim \|\phi\|_{L^p_\omega}^p, \quad \phi \in L^p_\omega.$$

- This can be rewritten as

$$\int_0^1 \left(U(r) \int_r^1 \psi(s)\, ds \right)^p dr \lesssim \int_0^1 \psi^p(r) V^p(r), \quad \psi \in L^p_{V^p}, \tag{1}$$

where

$$U(r) = \begin{cases} K(r)\omega(r)^{1/p}, & 0 \le r < 1 \\ 0, & r \ge 1 \end{cases},$$

and

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- But (1) is equivalent to

$$\sup_{0<r<1} \left(\int_0^r K^p(s)\omega(s)\, ds \right) \widehat{\omega}(r)^{\frac{p}{p'}} < \infty \tag{2}$$

by a result on Hardy operators due to Muckenhoupt (Studia Math. 1972).

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- Two applications of the Bernoulli-l'Hôpital theorem now give

$$\liminf_{r\to 1^-} \frac{\int_0^r K^p(s)\omega(s)\, ds}{\widehat{\omega}(r)^{-\frac{p}{p'}}} \ge \frac{1}{p-1} \liminf_{r\to 1^-} \left(\frac{\psi_\omega(r)}{1-r} \right)^p = \infty.$$

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- Therefore (2) is false and consequently, $P^+_\omega : L^p_\omega \to L^p_\omega$ is not bounded.