Two weight norm inequalities for fractional integrals and commutators

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Lecture 3: A_p bump conditions

Two weight $A_{p,q}$ condition

Given $1 < p \leq q < \infty$ and $0 < \alpha < n$, $(u,\sigma) \in A^{\alpha}_{p,q}$ if

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_Q u\,dx \right)^{\frac{1}{q}} \left(\fint_Q \sigma\,dx \right)^{\frac{1}{p'}} < \infty.$$

If $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, and $u = w^q$, $\sigma = w^{-p'}$, this becomes the one-weight $A_{p,q}$ condition.

Characterization of the weak type for M_α

Theorem

Given $1 < p \leq q < \infty$ and $0 < \alpha < n$, then $(u,\sigma) \in A^{\alpha}_{p,q}$ if and only if

$$M_\alpha(\cdot\sigma) : L^p(\sigma) \to L^{q,\infty}(u).$$

Implicit in Muckenhoupt-Wheeden (1974)

$A_{p,q}$ Condition not sufficient

Example (DCU-Moen 2013)

Given $1 < p \leq q < \infty$ and $0 < \alpha < n$, there exists a pair of weights (u, σ) that satisfy the two weight $A^{\alpha}_{p,q}$ condition but there exists $f \in L^p(\sigma)$ such that $M_\alpha(f\sigma) \notin L^q(u)$.

Folklore: may have been known earlier.

Factored weights

Lemma

If $1 < p \leq q < \infty$ and $0 < \alpha < n$, and given $w_1, w_2 \in L^1_{loc}$, then there exists $\gamma > 0$ such that if

$$u = w_1(M_\gamma w_2)^{-\frac{q}{p'}}, \quad \sigma = w_2(M_\gamma w_1)^{-\frac{p'}{q}},$$

then $(u, \sigma) \in A^{\alpha}_{p,q}$.

Factored weights systematically developed in DCU-Martell-Pérez (2011).

Sketch of counter-example

Define

$$E = \bigcup_{j \geq 0} [j, j + (j+1)^{-\gamma}), \qquad w_1 = \chi_E,$$

Then $M_\gamma w_1 \approx 1$

Let $f = w_2 = \chi_{[0,1]}$; then for $x \geq 2$,

$$M_\gamma w_2(x) \approx |x|^{\gamma-1}, \qquad M_\alpha(f\sigma)(x) \approx |x|^{\alpha-1}.$$

Bump conditions

- Generalize $A_{p,q}$ condition
- Universal sufficient conditions
- Easier to check than testing conditions
- Geometric condition on weights themselves
- Works well with CZ cubes

Introduced by Neugebauer (1983);
Systematically developed by Pérez (1994+)

Bumping the A_p condition

Rewrite $A^{\alpha}_{p,q}$ condition:

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \|u^{\frac{1}{q}}\|_{q,Q} \|\sigma^{\frac{1}{p'}}\|_{p',Q} < \infty$$

Key idea: replace localized L^q, $L^{p'}$ norms with larger norms in the scale of Orlicz spaces.

Bump functions

A Young function $B : [0,\infty) \to [0,\infty)$ is continuous, convex, increasing, $B(0) = 0$, and $B(t)/t \to \infty$ as $t \to \infty$.

Associate Young function $\bar{B}$

$$B^{-1}(t)\bar{B}^{-1}(t) \approx t$$

Key example: log-bumps

$$B(t) = t^p \log(e+t)^{p-1+\delta}, \qquad \bar{B}(t) \approx \frac{t^{p'}}{\log(e+t)^{1+(p'-1)\delta}}$$

Orlicz norms

Luxemburg norm: given a Young function B

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \fint_Q B\left(\frac{|f(x)|}{\lambda}\right)\,dx \le 1\right\}.$$

Hölder's inequality:

$$|\langle fg\rangle_Q| \le \fint_Q |f(x)g(x)|\,dx \le 2\|f\|_{B,Q}\|g\|_{\bar{B},Q}.$$

Orlicz maximal operators

Given Young function B, define

$$M_B f(x) = \sup_Q \|f\|_{B,Q}\,\chi_Q(x).$$

Theorem (Pérez 1995)

Given a Young function B, $M_B : L^p \to L^p$, if and only if $B \in B_p$:

$$\int_1^\infty \frac{B(t)}{t^p}\frac{dt}{t} < \infty.$$

For necessity, see Liu-Luque (2014).

The size of bumps

If $B \in B_p$, then $B(t) \lesssim t^p \qquad \bar{B}(t) \gtrsim t^{p'}$.

N.B. $B(t) = t^p$ not in B_p

Bumps for M_α and I_α

Theorem (Pérez (1994))

Given $1 < p \leq q < \infty$ and weights (u, σ), if $\bar{B} \in B_p$ and

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \|u^{\frac{1}{q}}\|_{q,Q} \|\sigma^{\frac{1}{p'}}\|_{B,Q} < \infty,$$

Then $M_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(u)$.

If $\bar{A} \in B_{q'}$, $\bar{B} \in B_p$ and

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \|u^{\frac{1}{q}}\|_{A,Q} \|\sigma^{\frac{1}{p'}}\|_{B,Q} < \infty,$$

Then $I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(u)$.

Proof for I_α, $p = q$

- Use sparse dyadic operator
- Apply duality
- Hölder's inequality to separate functions and weights
- Bump condition and Orlicz maximal operators to evaluate sum

Proof for I_α, $p = q$

$$\begin{aligned}
\int_{\mathbb{R}^n} & I_\alpha^{\mathcal{S}}(f\sigma)\, gu\, dx \\
&= \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} \langle f\sigma\rangle_Q \langle gu\rangle_Q |Q| \\
&\lesssim \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} \|f\sigma^{\frac{1}{p}}\|_{\bar{B},Q} \|\sigma^{\frac{1}{p'}}\|_{B,Q} \|gu^{\frac{1}{p'}}\|_{\bar{A},Q} \|u^{\frac{1}{p}}\|_{A,Q} |Q| \\
&\lesssim \sum_{Q\in\mathcal{S}} \|f\sigma^{\frac{1}{p}}\|_{\bar{B},Q} \|gu^{\frac{1}{p'}}\|_{\bar{A},Q} |E(Q)| \\
&\lesssim \sum_{Q\in\mathcal{S}} \int_{E(Q)} M_{\bar{B}}(f\sigma^{\frac{1}{p}}) M_{\bar{A}}(gu^{\frac{1}{p'}})\, dx \\
&\lesssim \|M_{\bar{B}}(f\sigma^{\frac{1}{p}})\|_p \|M_{\bar{A}}(gu^{\frac{1}{p'}})\|_{p'} \\
&\lesssim \|f\|_{L^{p}(\sigma)} \|g\|_{L^{p'}(u)}
\end{aligned}$$

Conjoined bumps for commutators

Theorem (DCU-Moen 2012)

Given $b \in BMO$ and $1 < p \le q < \infty$, if (u, σ) satisfies

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \|u^{\frac{1}{q}}\|_{A,Q} \|\sigma^{\frac{1}{p'}}\|_{B,Q} < \infty,$$

where

$$A(t) = t^q \log(e+t)^{2q-1+\delta}, \ B(t) = t^{p'} \log(e+t)^{2p'-1+\delta},$$

then

$$[b, I_\alpha](\cdot\sigma) : L^p(\sigma) \to L^q(u).$$

Sketch of proof I

Prove for dyadic operator

$$J_{\alpha,b}(f\sigma)(x) = \sum_{Q\in\mathcal{D}} |Q|^{\frac{\alpha}{n}} \fint_Q |b(x) - b(y)| f(y)\sigma(y)\, dy \,\chi_Q(x)$$

Use duality and split sum

$$\begin{aligned} &\int_{\mathbb{R}^n} J_{\alpha,b}(f\sigma) gu\, dx \\ &\le \sum_{Q\in\mathcal{D}} |Q|^{\frac{\alpha}{n}} \fint_Q |b - \langle b\rangle_Q| f\sigma\, dy \langle gu\rangle_Q |Q| \\ &\quad + \sum_{Q\in\mathcal{D}} |Q|^{\frac{\alpha}{n}} \fint_Q |b - \langle b\rangle_Q| gu\, dx \langle f\sigma\rangle_Q |Q| \end{aligned}$$

Sketch of proof II

By symmetry, estimate first sum. Restrict to cubes contained in Q_0.

Use corona decomposition of $|b - \langle b\rangle_Q| f\sigma$ wrt dx:

$$\begin{aligned} &\sum_{F\in\mathcal{F}} \fint_F |b - \langle b\rangle_F| f\sigma\, dy \sum_{Q\subset F} |Q|^{\frac{\alpha}{n}} \int_Q gu\, dx \\ &\le \sum_{F\in\mathcal{F}} |F|^{\frac{\alpha}{n}} \|b\|_{\exp L,F} \|f\sigma\|_{L\log L,F} \int_F gu\, dx \\ &\lesssim \|b\|_{BMO} \sum_{F\in\mathcal{F}} |F|^{\frac{\alpha}{n}} \|f\sigma\|_{L\log L,F} \fint_F gu\, dx |E(F)| \end{aligned}$$

Use generalized Hölder's inequality and bump condition to continue as for I_α.

Weak type inequalities

What are the correct bump conditions for weak type inequalities?

How are these related to bump conditions for strong type inequalities?

Muckenhoupt-Wheeden conjectures

Given $1 < p \leq q < \infty$, $0 < \alpha < n$ and (u, σ),

if

$$M_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(u), \qquad (M)$$
$$M_\alpha(\cdot u) : L^{q'}(u) \to L^{p'}(\sigma), \qquad (M^*)$$

then $I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(u)$.

If (M^*) holds, then

$$I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^{q,\infty}(u).$$

Off-diagonal results

Theorem (DCU-Moen 2013)

Given $1 < p < q < \infty$, $0 < \alpha < n$ and (u, σ), if (M) and (M^*) hold, then

$$I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(u).$$

If (M^*) holds, then

$$I_\alpha(\cdot\sigma) : L^p(\sigma) \to L^{q,\infty}(u).$$

Proof (easy!)

Recall off-diagonal testing conditions:

$$\left(\int_Q I^{\mathcal{D},+}_{\alpha,Q}(\sigma\chi_Q)^q u\,dx\right)^{\frac{1}{q}} \leq M_1 \left(\int_Q \sigma\,dx\right)^{\frac{1}{p}} \qquad (T_+)$$

$$\left(\int_Q I^{\mathcal{D},+}_{\alpha,Q}(u\chi_Q)^{p'} \sigma\,dx\right)^{\frac{1}{p'}} \leq M_2 \left(\int_Q u\,dx\right)^{1/q'} \qquad (T_+^*)$$

where

$$I^{\mathcal{D},+}_{\alpha,Q} f(x) = \sum_{\substack{Q'\in\mathcal{D}\\ Q\subsetneq Q'}} |Q'|^{\frac{\alpha}{n}} \langle f\rangle_{Q'}\, \chi_{Q'}(x).$$

Restating the condition

Summing the geometric series:

$$I^{\mathcal{D},+}_{\alpha,Q}(\sigma\chi_Q)(x) = \sum_{\substack{Q'\in\mathcal{D}\\ Q\subsetneq Q'}} |Q'|^{\frac{\alpha}{n}} \langle \sigma\chi_Q\rangle_{Q'}\, \chi_{Q'}(x)$$
$$\leq |Q|^{1-\frac{\alpha}{n}} \sum_{\substack{Q'\in\mathcal{D}\\ Q\subsetneq Q'}} |Q'|^{\frac{\alpha}{n}-1} M^{\mathcal{D}}_\alpha(\sigma\chi_Q)(x) \leq C M^{\mathcal{D}}_\alpha(\sigma\chi_Q)(x)$$

Condition (T_+) becomes

$$\left(\int_Q M^{\mathcal{D}}_\alpha(\sigma\chi_Q)^q u\,dx\right)^{\frac{1}{q}} \leq M_1 \left(\int_Q \sigma\,dx\right)^{\frac{1}{p}} \qquad (MT)$$

Separated bump condition

Testing conditions implied by:

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \|u^{\frac{1}{q}}\|_{A,Q} \|\sigma^{\frac{1}{p'}}\|_{p',Q} < \infty, \quad \bar{A} \in B_{q'} \quad (BL)$$

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \|u^{\frac{1}{q}}\|_{q,Q} \|\sigma^{\frac{1}{p'}}\|_{B,Q} < \infty, \quad \bar{B} \in B_{p} \quad (BR)$$

This suggests that two bump conditions are sufficient for strong type inequalities
and the dual bump condition (BL) should be sufficient for weak type inequalities.

A weaker condition

Example (Anderson-DCU-Moen (2013))

Given $1 < p \leq q < \infty$ and $0 < \alpha <$, there exists (u, σ) and Young functions A, B with $\bar{A} \in B_{q'}$, $\bar{B} \in B_p$, such that (u, σ) satisfy the separated bump conditions but not the conjoined bump condition.

Example for $p = q = 2$, $\alpha = 0$, but easily modified.

Two questions when $p = q$

- Are the MW conjectures true: do (M) and (M^*) imply $I_\alpha(\cdot\sigma) : L^p(\sigma) \to {}^{p}(u)$?

  Conjecture: no.

- Do the separated bump conditions (BL) and (BR) imply strong and weak type inequalities?

  Conjecture: it depends on the size of the bump.

Best known result

Theorem (DCU-Martell-Pérez (2011))

Given $1 < p < \infty$, $0 < \alpha < n$ and (u, σ), if (ML) and (MR) hold with

$$A(t) = t^p \log(e+t)^{2p-1+\delta}, \quad B(t) = t^{p'} \log(e+t)^{2p'-1+\delta},$$

then $I_\alpha(\cdot\sigma) : L^p(\sigma) \to {}^{p}(u)$.

If (ML) holds, then $I_\alpha(\cdot\sigma) : L^p(\sigma) \to {}^{p,\infty}(u)$.

Idea of proof

Theorem follows from two weight extrapolation and weak type inequality:

$$u(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| M_{B,\alpha} u(x)\, dx,$$

where

$$M_{B,\alpha} u(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \|u\|_{B,Q}\, \chi_Q(x), \qquad B(t) = t \log(e+t)^{1+\epsilon}.$$

Three final questions

- Can you prove this result using testing conditions and the corona decomposition?
- Can you prove this result for log bumps:

  $$A(t) = t^p \log(e+t)^{p-1+\delta}, \; B(t) = t^{p'} \log(e+t)^{p'-1+\delta}?$$

- can you prove weak $(1,1)$ inequality with $B(t) = t \log(e+t)^{\epsilon}$?

Very recent work by Lacey (2014) and Treil and Volberg (2014) suggests even weaker conditions are possible, but not general B_p bumps.

End of Lecture 3
Thank you very much! Muchas gracias!