Limited Rubio de Francia extrapolation results with applications to Bochner-Riesz operators

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Introduction:

The Bochner-Riesz operators in $\mathbb{R}^d$ are defined as

$$\widehat{T_\beta f}\ (\xi) = \left(1 - |\xi|^2\right)_+^\beta \widehat{f}(\xi)$$

with $\beta > 0$ and $t_+ = \max(t, 0)$.

If $\beta = 0$, this operator is the Fourier multiplier of the unit ball and it is well-known that T_0 is bounded on L^p if and only if $p = 2$ (Fefferman, 1971).

The maximal Bochner-Riesz operator is defined by

$$T_\beta^* f(x) = \sup_{r>0} |T_\beta^r f(x)|$$

where

$$\widehat{T_\beta^r f}\ (\xi) = \left(1 - \frac{|\xi|^2}{r^2}\right)_+^\beta \widehat{f}(\xi)\cdot$$

If $\beta > \frac{d-1}{2}$, $T_\beta^* f$ is pointwise majorized by the Hardy-Littlewood maximal operator.

For β below the critical index the study of the boundedness properties of the Bochner-Riesz operators constitutes an active area of research.

1) If T_β is bounded on L^p then necessarily

$$\frac{2d}{d+1+2\beta} < p < \frac{2d}{d-1-2\beta}.$$

2) The **Bochner-Riesz conjecture** states that if $p > 1$ and

$$\beta > \beta(p) = \max\left(d\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right)$$

then T_β is bounded in L^p.

3) This conjecture was proved in two dimensions by Carleson and Sjölin in 1972, but is still open in higher dimensions.

4) For $n = 3$, T. Wolff proved in 1995 the conjecture for $p \leq 42/31$ and Tao and Vargas in 2000 for $p \geq 4 - 2/7$.

5) In 2004, Lee proved the conjecture for $p > (2n+4)/n$ or $p < (2n+4)/(n+4)$.

Weighted theory: Muckenhoupt weights.

Let w be a weight in the Muckenhoupt class A_p; that is

$$||w||_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{p-1} < \infty,$$

and

$$||w||_{A_1} = \text{ess sup} \frac{Mw(x)}{w(x)} < \infty,$$

with

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

the Hardy-Littlewood maximal operator.

Then, it is well known (Muckenhoupt 1972, TAMS) that if $p > 1$,

$$M : L^p(w) \to L^p(w) \iff w \in A_p$$

and

$$M : L^1(w) \to L^{1,\infty}(w) \iff w \in A_1$$

Also, if T is a Calderón-Zygmund operator (Hilbert transform)

$$T : L^p(w) \to L^p(w) \qquad \text{if} \qquad w \in A_p$$

and

$$\boxed{T : L^1(w) \to L^{1,\infty}(w) \qquad \text{if} \qquad w \in A_1}$$

Rubio de Francia's extrapolation theorem

Theorem 1

If for some $1 \leq p_0 < \infty$ *and for every* $w \in A_{p_0}$,

$$T : L^{p_0}(w) \to L^{p_0}(w)$$

then, for every $1 < p < \infty$ *and every* $w \in A_p$,

$$T : L^p(w) \to L^p(w).$$

Fundamental remark:

If $\beta < \frac{d-1}{2}$, T_β can not be bounded on $L^p(w)$ for every $w \in A_p$ (for any p).

However, the following results are known:

Theorem 2

i) (A. Vargas, 1996)

If $\beta = (d-1)/2$ *and* $w \in A_1$,

$$T_\beta^* : L^1(w) \longrightarrow L^{1,\infty}(w)$$

is bounded.

ii) (M. Christ, 1985).

If $\frac{d-1}{2(d+1)} < \beta < \frac{d-1}{2}$, *then*

$$T_\beta^* : L^2(u) \longrightarrow L^2(u)$$

is bounded for every $u = u_0^s$ *such that* $u_0 \in A_1$ *and* $s < (1+2\beta)/d$.

iii)(Duoandikoetxea, Moyua, Oruetxebarria and Seijo, 2008).

If $0 < \beta < (d-1)/2$,

$$T_\beta^* : L^2(w) \longrightarrow L^2(w)$$

for every $w(x) = v(x)^{2\beta/d-1}$ *with* $v \in A_2(\mathbb{R}^d)$.

Related to the theory of "two weights" there are few results:

Theorem 3 *(Carbery-Seeger, 2000)*

For every w,

$$T_\beta : L^2(W) \longrightarrow L^2(w)$$

is bounded with $W = Sw$.

The main goal of this talk is to prove weighted estimates for the Bochner-Riesz operators of two kind:

1)

$$T_\beta : L^p(u) \longrightarrow L^{q,\infty}(v)$$

and

2)

$$T_\beta : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w)$$

where $\Lambda^p(w)$ is the weighted Lorentz space.

Since, every weight $v \in A_p$ can be factorized by

$$v = v_0 v_1^{1-p}, \qquad v_j \in A_1,$$

we can easily see that the two last results (Christ and DMOS) in the above theorem can be written by

$$T_\beta^* : L^2(u) \longrightarrow L^2(u)$$

with $u = u_0^{\alpha_0} u_1^{-\alpha_1}$ for certain values of $\alpha_j \in [0, 1]$.

Definition 1

Given $0 \leq \alpha_0 \leq 1$ *and* $0 \leq \alpha_1 \leq 1$, *let us define the class*

$$A_{p;(\alpha_0,\alpha_1)} = \left\{ w = w_0^{\alpha_0} w_1^{\alpha_1(1-p)}; w_j \in A_1 \right\}.$$

Remarks 4

i) $A_{p;(1,1)} = A_p$.

ii) $A_{p;(\alpha_0,\alpha_1)} \subset A_p$ *since, given* $u \in A_1$, $u^s \in A_1$ *for every* $0 < s < 1$.

iii) If $1 \leq q < p$, *then* $A_q = A_{p;(1,\frac{q-1}{p-1})}$.

iv) If $\alpha_0 = \alpha_1 = \alpha$, $w \in A_{p;(\alpha,\alpha)}$ *if and only if* $w^{1/\alpha} \in A_p$.

v) For every $p \geq 1$ *and every* $\alpha_j \in (0,1)$, $1 \in A_{p;(\alpha_0,\alpha_1)}$.

Theorem 5

Let T be such that, for some $1 \leq p < \infty$

$$T : L^p(w) \longrightarrow L^{p,\infty}(w)$$

for every $w \in A_{p;(\alpha_0,\alpha_1)}$ with constant $C_{\|w\|_{A_p}}$. Then, let

$$p_0' = \frac{p'}{1-\alpha_1} \quad \textit{and} \quad p_1 = \frac{p}{1-\alpha_0}. \tag{1}$$

Then, given $p_0 \leq s \leq p$, $0 < \delta < 1/s$ and $0 < \alpha < \alpha_0$, it holds that, for every measurable function $v > 0$ and every weight u,

$$\lambda^u_{Tf}(y) \lesssim \lambda^u_{M_\delta(fv^{-1})v}(y) + \frac{1}{y^s}\int_{\mathbb{R}^n} f^s(x)v(x)^{p-s}M_\alpha(v^{s-p}u\chi_{\{|Tf|>y\}})(x)dx, \tag{2}$$

where

$$M_\mu f(x) = M(|f|^{1/\mu})^\mu(x).$$

Corollary 6

Let T be as in the previous theorem.

Then, for every $0 < \delta < 1/p$ and $0 < \alpha < \alpha_0$, it holds that, for every weight u,

$$\lambda^u_{Tf}(y) \lesssim \lambda^u_{M_\delta f}(y) + \frac{1}{y^p} \int_{\mathbb{R}^n} f^p(x) M_\alpha(u\chi_{\{|Tf|>y\}})(x)dx, \tag{3}$$

Remark 1 *In fact, this formula in the case $\alpha_0 = \alpha_1 = 1$ was proved in a previous paper with J. Soria and R. Torres.*

Definition 2

We say that an operator T is a limited RF-operator of type $(p; \alpha_0, \alpha_1)$ if

$$T : L^p(u) \longrightarrow L^p(u)$$

is bounded for every $u \in A_{p;(\alpha_0,\alpha_1)}$ with constant depending on $\|u\|_{A_p}$.

Consequences

(I) The first main application of (3) is the following extrapolation result:

Theorem 7

Let T be a limited RF-operator of type $(p;\alpha_0,\alpha_1)$ and let $q \in (p_0,p_1)$. Then,

$$T : L^q(u) \longrightarrow L^{q,\infty}(u)$$

is bounded, for every

$$u \in A_{q;(\alpha_0(q),\alpha_1(q))},$$

where

$$p_0' = \frac{q'}{1-\alpha_1(q)} \quad \text{and} \quad p_1 = \frac{q}{1-\alpha_0(q)}.$$

(II) Two weights estimates:

An interesting application of the above distribution formulas is that from them we can deduce conditions on a pair of weights in order to have two weights estimates.

Theorem 8

Let T be a limited RF-operator of type $(p; \alpha_0, \alpha_1)$ for some $p \geq 1$. Let $p_0 < q < p_1$ and let u and w be two weights. If there exist $s > 1$, $r > 1$ and $\alpha < \alpha_0(q)$ satisfying that

$$M : L^{\alpha s}(u^{1-s}) \longrightarrow L^{\alpha r}(w^{1-r}) \tag{4}$$

is bounded, then

$$T : L^{r'q}(w) \longrightarrow L^{s'q,\infty}(u)$$

is bounded.

Theorem 9

Let T be a limited RF-operator of type $(p;\alpha_0,\alpha_1)$ for some $p \geq 1$ and let $p_0 < q < p_1$. If for some $\alpha < 1-\frac{q}{p_1}$ and $\delta < 1/p_0$,

$$M : L^{\delta q}(w) \longrightarrow L^{\delta q,\infty}(u) \tag{5}$$

and

$$M : L^{\alpha\frac{q}{q-p_0}}(u^{\frac{-p_0}{q-p_0}}) \longrightarrow L^{\alpha\frac{q}{q-p_0}}(w^{\frac{-p_0}{q-p_0}}) \tag{6}$$

are bounded, then

$$T : L^{q}(w) \longrightarrow L^{q,\infty}(u)$$

is bounded.

(III) Estimates in r.i. spaces:

If we take $u = v = 1$ in (3), use Hardy's inequality and the fact that

$$(M_\alpha(\chi_{\{g>y\}}))^*(t) \approx \min\left(\frac{\lambda_g(y)}{t}, 1\right)^\alpha,$$

we have the following formulas which are very useful to prove boundedness on r.i. spaces.

Theorem 10

Let T be a limited RF-operator of type $(p; \alpha_0, \alpha_1)$ for some $p \geq 1$. If $p_0 < s < p_1$ and $\alpha < 1 - \frac{s}{p_1}$, then

$$\lambda_{Tf}(y) \lesssim \frac{1}{y^s} \int_0^\infty f^*(t)^s \min\left(\frac{\lambda_{Tf}(y)}{t}, 1\right)^\alpha dt. \tag{7}$$

Boundedness on weighted Lorentz spaces:

Let us recall that the weighted Lorentz spaces $\Lambda^q(w)$ were introduced by Lorentz in 1951 and are defined by the condition

$$\|f\|_{\Lambda^q(w)} = \left(\int_0^\infty f^*(t)^q w(t) dt \right)^{1/q} < \infty,$$

where f^* is the decreasing rearrangement of f

$$f^*(t) = \inf\{s > 0; \lambda_f(s) > t\}$$

Also, the weak version of these spaces are defined by the condition

$$\|f\|_{\Lambda^{q,\infty}(w)} = \sup_{t>0} f^*(t) W(t)^{1/q} < \infty,$$

where $W(t) = \int_0^t w(s) ds$.

Theorem 11

Let T be a limited RF-operator of type $(p;\alpha_0,\alpha_1)$ for some $p\geq 1$ and let $q\leq p_0$.

Then if w is a weight in $(0,\infty)$ satisfying

i) there exists $\beta<\frac{1}{p_0}$ such that $\frac{W(t)^{1/q}}{t^\beta}$ is a decreasing function and

ii) there exists $\gamma>\frac{1}{p_1}$ such that $\frac{W(t)^{1/q}}{t^\gamma}$ is an increasing function, then

$$T:\Lambda^q(w)\longrightarrow\Lambda^{q,\infty}(w)$$

is bounded.

A) From Theorem 2, ii) we have that if

$$\frac{d-1}{2(d+1)} < \beta < \frac{d-1}{2}, \tag{8}$$

T_β is a limited RF-operator of type $(2; \alpha_0, \alpha_1)$ with

$$\alpha_0 < \frac{1+2\beta}{d} \qquad \text{and} \qquad \alpha_1 = 0.$$

Then

$$p_0 = 2 \qquad \text{and} \qquad p_1 < \frac{2d}{d-1-2\beta}.$$

Consequently, we get the following result due to Fefferman-Stein:

Corollary 12

If $p > \frac{2(d+1)}{d-1}$, T_β is bounded on L^p for every

$$\beta > \beta(p) = \max\left(d\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right).$$

By duality, T_β a limited RF-operator of type $(2; \alpha_0, \alpha_1)$ with

$$\alpha_0 = 0 \qquad \text{and} \qquad \alpha_1 < \frac{1+2\beta}{d}.$$

Then

$$\frac{2d}{d+1+2\beta} < p_0 \qquad \text{and} \qquad p_1 = 2.$$

Consequently, we get the following result:

Corollary 13

If $p < \frac{2(d+1)}{d+3}$, T_β is bounded on L^p for every

$$\beta > \beta(p) = \max\left(d\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right).$$

B) Also, from Theorem 2, iii), for every $\beta > 0$, T_β is limited RF-operator of type $(2; \alpha_0, \alpha_1)$ with

$$\alpha_0 = \alpha_1 = \frac{2\beta}{d-1}$$

and hence

$$p_0 = \frac{2(d-1)}{d-1+2\beta}, \qquad p_1 = \frac{2(d-1)}{d-1-2\beta}.$$

Corollary 14

If $\beta > 0$, T_β *is bounded in* L^p *for every*

$$\frac{2(d-1)}{d-1+2\beta} < p < \frac{2(d-1)}{d-1-2\beta}.$$

We have to mention here the result of Lee in 2004 since his result is better than the one presented in our corollaries. However, the easy proof of the above particular case makes interesting its presentation.

Corollary 15

Let $d = 3$. If there exist $\nu, \rho < 1 + \beta$ such that

$$M : L^{\nu}(w) \longrightarrow L^{\nu,\infty}(u)$$

and

$$M : L^{\rho}(u^{-1/\beta}) \longrightarrow L^{\rho}(w^{-1/\beta})$$

are bounded, we obtain that

$$T_{\beta} : L^{2}(w) \longrightarrow L^{2,\infty}(u)$$

is bounded.

Corollary 16

Let $d = 3$.

a) If $\beta > 1/4$ and there exist $r < 1/2$ and $s > \frac{1-\beta}{3}$ such that $W(t)t^{-r}$ is decreasing and $W(t)t^{-s}$ is increasing, then

$$T_\beta : \Lambda^1(w) \longrightarrow \Lambda^{1,\infty}(w)$$

is bounded.

b) If $\beta > 0$ and there exist $r < \frac{1+\beta}{2}$ and $s > \frac{1-\beta}{2}$ such that $W(t)t^{-r}$ is decreasing and $W(t)t^{-s}$ is increasing, then

$$T_\beta : \Lambda^1(w) \longrightarrow \Lambda^{1,\infty}(w)$$

is bounded.

FINAL REMARK:

All the previous results can be extended to the classes A_p^+ and A_p^- in order to cover similar results for the so-called one-sided operators.