Restriction estimates for some surfaces with vanishing curvatures

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Restriction of the Fourier Transform to Hypersurfaces

$K \subset \mathbb{R}^n$ compact set

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By duality, the restriction inequality is equivalent to

$$\|\widehat{fd\sigma}\|_{L^q(\mathbb{R}^{n+1})} \leq C\|f\|_{L^p(K \subset \mathbb{R}^n)}.$$

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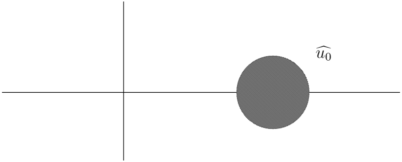
Ikromov–Kempe–Müller : Sharp result for finite type surfaces

The Knapp counterexample



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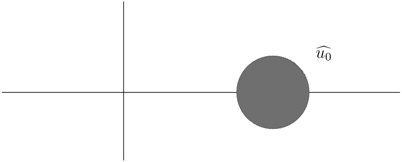

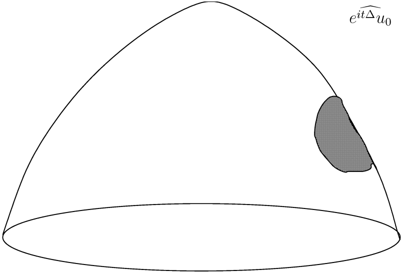

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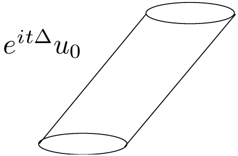

Bilinear restriction theorems

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Definition. Let $\pm\mathbf{N}(\xi) \in \mathbb{S}^n$ be the unit normal vector of S at ξ. We say that S_1 and S_2 are transversal if, for $\xi_1 \in S_1$ and $\xi_2 \in S_2$,

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Sharp result for n non-vanishing positive curvatures Tao (2003)

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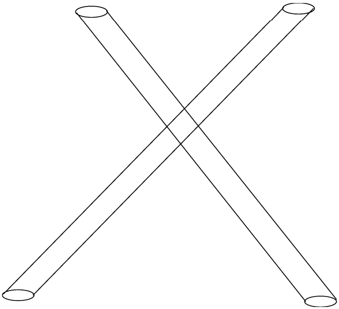

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Best result for cones ($n-1$ non-vanishing positive curvatures) Wolff (2001), Tao (2001)
Best result for $n-1$ non-vanishing curvatures, different signs Lee (2006)

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Third transversality condition, equivalent definition : Set

$$\Gamma_2 = \{t\mathbf{N}^2(\xi) : \xi \in \Pi_{z_1,z_2}, 1 \le |t| \le 2\}.$$

The third condition equivalently means that any normal vector $\mathbf{N}^1$ of S_1 is transversal to Γ_2 plus a similar condition for $\mathbf{N}^2$ and Γ_1.)

For the positively curved surfaces (e.g. the cone, sphere, or paraboloid) the third transversality can be obtained from the first separation condition. This is actually the separation condition which was used to obtain the best possible bilinear restriction estimates for the case of nonvanishing curvatures, different signs (Lee, V).

Theorem

Let $1 \leq k \leq n-1$. Suppose that S is a smooth compact surface in $\mathbb{R}^{n+1}$ with k-nonvanishing curvatures. If the surfaces $S_1, S_2 \subset S$ satisfy the three transversality conditions, then for $p > \frac{k+4}{k+2}$

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If we assume only the first and third orthogonality conditions, we prove the estimate for $p > \frac{k+3}{k+1}$. This exponent is sharp.

The model examples

Let us set

$$Q = \{(\xi, \eta, \rho, \delta) : |\xi| \le 1,\ |\eta| \le 1,\ \rho \in [1,2],\ \delta \in [1,2]\} \subset \mathbb{R}^4.$$

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Note that

$$\phi(\xi, \rho) = \frac{|\xi|^2}{\rho}$$

defines the usual cone.

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For fixed q, the result is sharp.

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$$\mathbf{N}(\xi, \eta, \rho, \delta) \sim \left(2\tfrac{\xi}{\rho}, -2\tfrac{\eta}{\delta}, -\tfrac{\xi^2}{\rho^2}, \tfrac{\eta^2}{\delta^2}, -1\right) = \left(2\theta, -2\lambda, -\theta^2, \lambda^2, -1\right)$$

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Orthogonality condition 1 :

$\mathbf{N}(\xi_1,\eta_1,\rho_1,\delta_1)$ and $\mathbf{N}(\xi_2,\eta_2,\rho_2,\delta_2)$ are not paralell.

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Second transversality condition :

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Finally,

$$< \nabla\phi(u) - \nabla\phi(u+w), \mathcal{N}_1 >= |\frac{\xi_1}{\rho_1} - \frac{\xi_2}{\rho_2}|^2.$$

Third transversality condition :
Set
$$\Gamma_2 = \{t\mathbf{N}^2(\xi) : \xi \in \Pi_{z_1,z_2}, 1 \leq |t| \leq 2\}.$$
Any normal vector $\mathbf{N}^1$ is transversal to Γ_2.

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