Multiresolution in $H^2(T)$ generated by a special Malmquist-Takenaka System

Margit Pap,
Marie Curie Fellow, NuHAG Univ. Vienna
University of Pécs, Hungary [1]
papmargit@citromail.hu, margit.pap@univie.ac.at

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- Motivation

- The continuous voice transform

- The voice transform of the Blaschke group

- Multiresolution analysis of $\mathcal{H}^2(\mathbb{T})$

- The projection operator to the n-th resolution level

- Reconstruction algorithm

Motivation

- In signal processing the rational orthogonal bases (Laguerre, Kautz and Malmquist-Takenaka systems) are more efficient.
- The successful application of rational orthogonal bases needs a priori knowledge of the poles of the transfer function that may cause a drawback of the method.
- We give a set of poles and using them we will generate a multiresolution in $H^2(\mathbb{T})$ and $H^2(\mathbb{D})$.
- The construction is an analogy with the discrete affine wavelets, and in fact is the discretization of the continuous voice transform generated by a representation of the Blaschke group over the space $H^2(\mathbb{T})$.

Totik's recovery theorem

Theorem

If $(z_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers in the open unit disc such that

$$\sum_{j=0}^{\infty}(1-|z_j|)=\infty,$$

then for all $f\in H^p(\mathbb{D})$ there are polynomials $p_{n,j}$ such that

$$\|f-\sum_{j=0}^{n}f(z_j)p_{n,j}\|_{H^p}\to 0,\ \text{ if } n\to\infty.$$

Recovery

- The coefficients of $p_{n,j}$ are given by integrals, which can not be determined exactly from $(f(z_j))_{j\in\ltimes}$.
- **Question:** How to give a recovery formula depending only on $(z_j)_{j\in\ltimes}$ and $(f(z_j))_{j\in\ltimes}$?
- **KEHE ZU, 1997** gave for $H^2(\mathbb{D})$ a possible algorithm of the recovery in generel for the set of uniqueness.
- In this talk I will present a special set of the points $(z_j)_{j\in\ltimes} \in \mathbb{D}$ which will be the base of the recovery using multiresolution in $H^2(\mathbb{D})$ and in $H^2(\mathbb{T})$.
- For this purpose we will need tools from non-commutative harmonic analysis over groups and the generalization of Fourier transform: the voice transform.

The continuous voice transform

- H. G. Feichtinger and K. H. Gröchenig unified the theory of Gábor and wavelet transforms into a single theory. The common generalization of these transforms is the so-called **voice transform**.
- In the construction of the voice-transform the starting point will be a locally compact topological group $(G, \cdot)$.

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- In the construction of the voice-transform the starting point will be a locally compact topological group $(G, \cdot)$.
- Let m be a left-invariant Haar measure of G:

$$\int_G f(x)\,dm(x) = \int_G f(a^{-1} \cdot x)\,dm(x), \quad (a \in G).$$

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Unitary representation

- **Unitary representation of the group** $(G,\cdot)$: Let us consider a Hilbert-space $(H,\langle\cdot,\cdot\rangle)$.
- $\mathcal{U}$ denote the set of unitary bijections $U: H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear operators which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$.

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- The homomorphism of the group $(G, \cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying

$$i) \quad U_{x \cdot y} = U_x \circ U_y \quad (x, y \in G),$$

$$ii) \quad G \ni x \to U_x f \in H \quad \text{is continuous for all } f \in H$$

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Affine wavelet transform

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The affine group:

$G = \{\ell_{(a,b)}(x) = ax + b : \mathbb{R} \to \mathbb{R} : (a, b) \in \mathbb{R}^* \times \mathbb{R}\}$

$\ell_1 \circ \ell_2(x) = a_1 a_2 x + a_1 b_2 + b_1, \ (a_1, b_1) \circ (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$

The representation of G on $L^2(\mathbb{R})$

$U_{(a,b)} f(x) = |a|^{-1/2} f(a^{-1}x - b)$

The affine wavelet transform is:

$W_\psi f(a, b) = |a|^{-1/2} \int_{\mathbb{R}} f(t)\overline{\psi(a^{-1}t - b)}dt = \langle f, U_{(a,b)}\psi\rangle.$

Discretization: Find a ψ such that

$$\psi_{n,k} = 2^{-n/2}\psi(2^{-n}x - k)$$

form a (orthonormal) basis in $L^2(\mathbb{R})$ which generate a

The Blaschke group

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$$B_a(z) := \epsilon \frac{z-b}{1-\bar{b}z} \quad (z \in \mathbb{C}, a = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, \bar{b}z \neq 1)$$

  the so called **Blaschke functions**,

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$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \ \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

- If $a \in \mathbb{B}$, then B_a is an 1-1 map on $\mathbb{T}$, $\mathbb{D}$ respectively.

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- The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group.

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The voice transform of the Blaschke group

- In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$.
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$$b = \frac{b_1 \overline{\epsilon}_2 + b_2}{1 + b_1 \overline{b_2} \overline{\epsilon}_2}, \quad \epsilon = \epsilon_1 \frac{\epsilon_2 + b_1 \overline{b_2}}{1 + \epsilon_2 \overline{b_1} b_2}.$$

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- The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \epsilon) \in \mathbb{B}$ is $a^{-1} = (-b\epsilon, \overline{\epsilon})$.

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$$\int_{\mathbb{B}} f(a)\, dm(a) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2}\, db_1 db_2 dt,$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

- Denote by $\epsilon_n(t) = e^{int}$ $(t \in \mathbb{I} = [0, 2\pi], n \in \mathbb{N})$, let consider the Hilbert space $H = H^2(\mathbb{T})$, the closure in $L^2(\mathbb{T})$-norm of the set

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The representation of the Blaschke group on $H^2(\mathbb{T})$

- The inner product is given by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{I}} f(e^{it})\overline{g(e^{it})}\, dt \quad (f, g \in H).$$

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$$(U_{a^{-1}} f)(z) := \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\overline{b}z)} f\left(\frac{e^{i\theta}(z-b)}{1-\overline{b}z}\right)$$

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$$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle \ \ (f, \rho \in H^2(\mathbb{T})),\ a = (re^{i\phi}, e^{i\psi}) \in \mathbb{B}.$$

- **Question:** How to choose a discrete subset $a_{k\ell} = (z_{k\ell}, 1) \in \mathbb{B}$ and $\rho \in H^2(\mathbb{T})$ such that the functions $U_{a_{k\ell}^{-1}}\rho$ generate a multiresolution decomposition in $H^2(\mathbb{T})$) and in $H^2(\mathbb{D})$) ?

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Pap M.: Hyperbolic Wavelets and Multiresolution in $H^2(\mathbb{T})$, Accepted for publication: Journal of Fourier Analysis and Applications DOI: 10.1007/s00041-011-9169-2

- Continuous voice transform:

$$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle \ \ (f, \rho \in H^2(\mathbb{T})),\ a = (re^{i\phi}, e^{i\psi}) \in \mathbb{B}.$$

- **Question:** How to choose a discrete subset $a_{k\ell} = (z_{k\ell}, 1) \in \mathbb{B}$ and $\rho \in H^2(\mathbb{T})$ such that the functions $U_{a_{k\ell}^{-1}}\rho$ generate a multiresolution decomposition in $H^2(\mathbb{T})$) and in $H^2(\mathbb{D})$) ?
- Let denote by $\mathbb{B}_1 = \left\{(r_k, 1) :\ r_k = \frac{2^k - 2^{-k}}{2^k + 2^{-k}},\ k \in \mathbb{Z}\right\}$.
- It can be proved that $(\mathbb{B}_1, \circ)$ is a subgroup of $(\mathbb{B}, \circ)$, and $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$

Multiresolution analysis of $L^2(\mathbb{R})$

Let $V_j,\ j \in \mathbb{Z}$ be a sequence of subspaces of $L^2(\mathbb{R})$. The collections of spaces $\{V_j,\ j \in \mathbb{Z}\}$ is called a multiresolution analysis with scaling function ϕ if the following conditions hold:

1. (nested) $V_j \subset V_{j+1}$

2. (density) $\overline{\cup V_j} = L^2(R)$

3. (separation) $\cap V_j = \{0\}$

4. (basis) The function ϕ belongs to V_0 and the set $\{2^{n/2}\phi(2^n x - k),\ k \in \mathbb{Z}\}$ is a (orthonormal) bases in V_n.

Multiresolution analysis of $\mathcal{H}^2(\mathbb{T})$

We want to give the analogue of the affine wavelet multiresolution analysis in $\mathcal{H}^2(\mathbb{T})$.

Definition

Let $V_j,\ j \in N$ be a sequence of subspaces of $H^2(T)$. The collections of spaces $\{V_j,\ j \in N\}$ is called a multiresolution if the following conditions hold:

1. *(nested)* $V_j \subset V_{j+1}$,
2. *(density)* $\overline{\cup V_j} = H^2(T)$
3. *(dilatation)* $U_{(r_1,1)^{-1}}(V_j) \subset V_{j+1}$
4. *(basis) There exist* $\psi_{j\ell}$ *(orthonormal) bases in* V_j.

Multiresolution analysis of $\mathcal{H}^2(\mathbb{T})$

Let us consider the set of points in the unit disc

$$A = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}},\ \ell = 0, 1, \ldots, 2^{2k} - 1,\ k = 0, 1, 2, \ldots\infty\},$$

$$A_k = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}},\ \ell \in \{0, 1, \ldots, 2^{2k} - 1\}\}.$$

A is not a Blaschke sequence: $\sum_{k,\ell}(1 - |z_{k\ell}|) = \infty$.

Multiresolution analysis of $\mathcal{H}^2(\mathbb{T})$

- Let us consider the function $p_0 = \varphi_{00} = 1,\ V_0 = \{c,\ \ c \in C\}$ and let

$$p_1(z) = U_{r_1^{-1}} p_0 = \frac{\sqrt{1 - r_1^2}}{(1 - r_1 z)},\ p_n(z) = (U_{r_1^{-1}} p_{n-1})(z) = \frac{\sqrt{1 - r_n^2}}{(1 - r_n z)},$$

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$$\varphi_{n,\ell}(z) = (U_{(r_{n-1} \circ r_1)^{-1}} p_0)(e^{i(t - \frac{2\pi\ell}{2^{2n}})}).$$

- Let us define the n-th resolution level by

$$V_n = \{f : D \to C,\ f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k,\ell}\varphi_{k,\ell},\ \ c_{k,\ell} \in C\ \}.$$

If a function $f \in V_n$, then $U_{(r_1)^{-1}} f \in V_{n+1}$.

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Multiresolution analysis of $\mathcal{H}^2(\mathbb{T})$

- The closed subset V_n is spanned by the nonorthogonal basis:

$$\{\varphi_{k,\ell},\ \ell = 0, 1, ..., 2^{2k} - 1,\ k = 1, ..., n\},$$

$$V_0 \subset V_1 \subset V_2 \subset .....V_n \subset ....\ H^2(T).$$

- Applying the Gram-Schmidt orthogonalization for this set of analytic linearly independent functions we obtain the Malmquist -Takenaka system corresponding to the set $\cup_{k=0}^{n} A_k$:

$$\psi_{m,\ell}(z) = \frac{\sqrt{1 - r_m^2}}{1 - \overline{z_{m\ell}} z} \prod_{k=0}^{m-1} \prod_{j=0}^{2^{2k}-1} \frac{z - z_{kj}}{1 - \overline{z_{kj}} z} \prod_{j'=0}^{\ell-1} \frac{z - z_{mj'}}{1 - \overline{z_{mj'}} z}.$$

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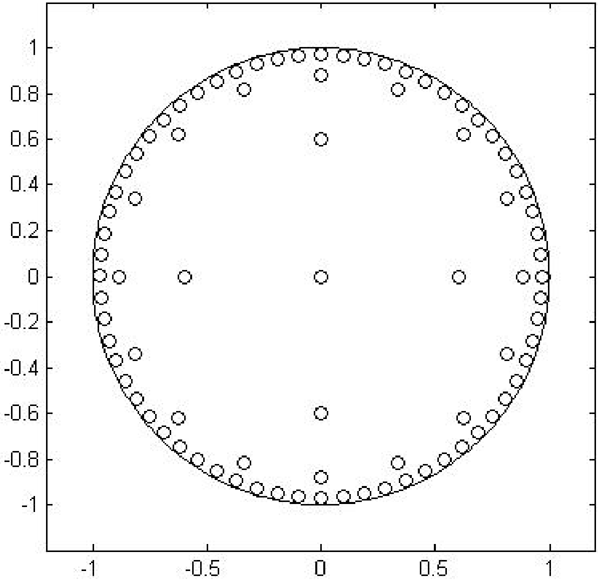

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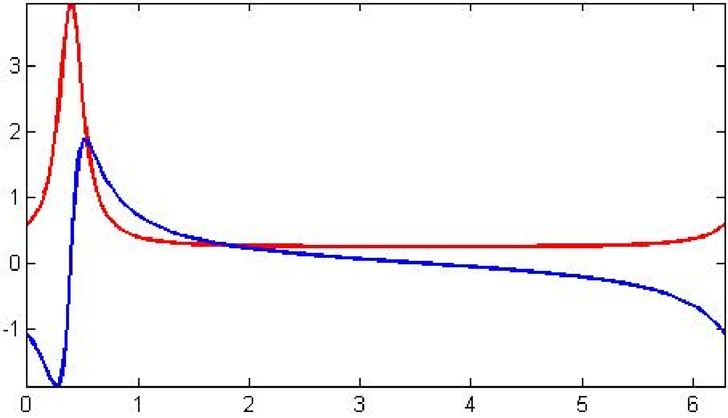
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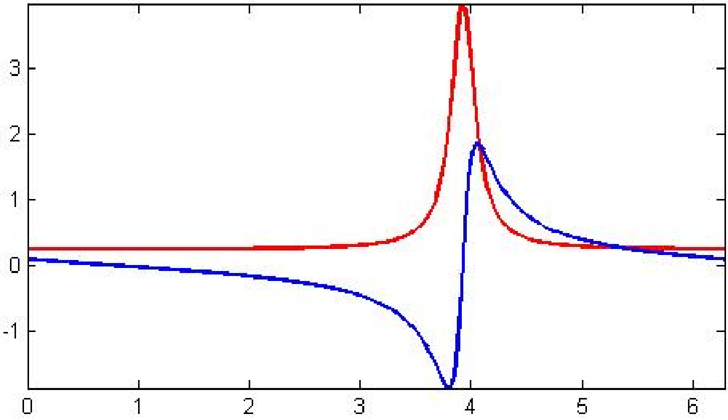

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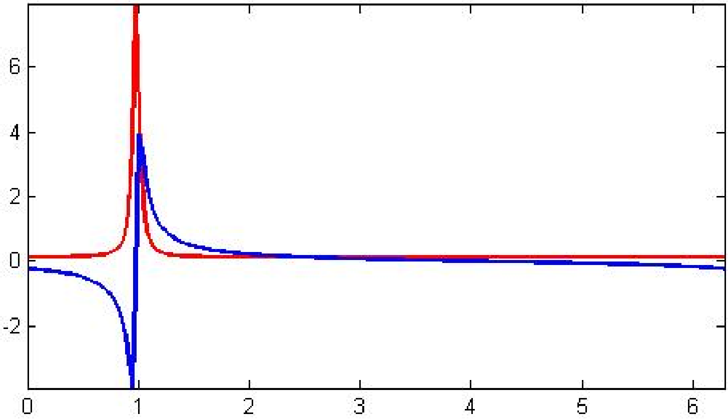
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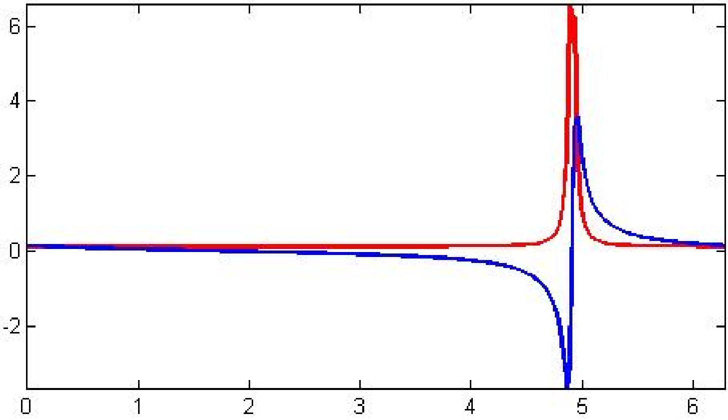

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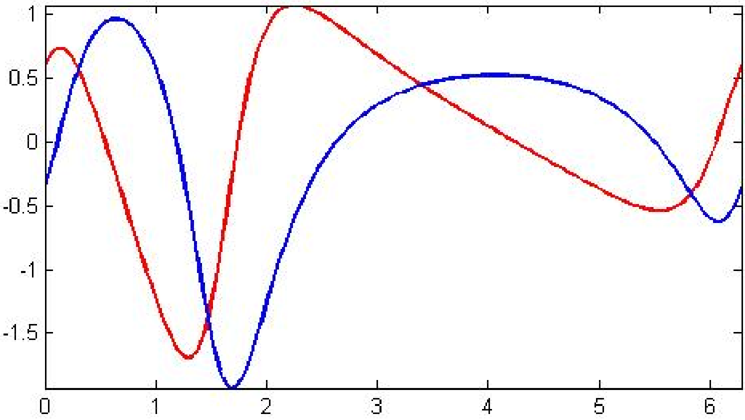
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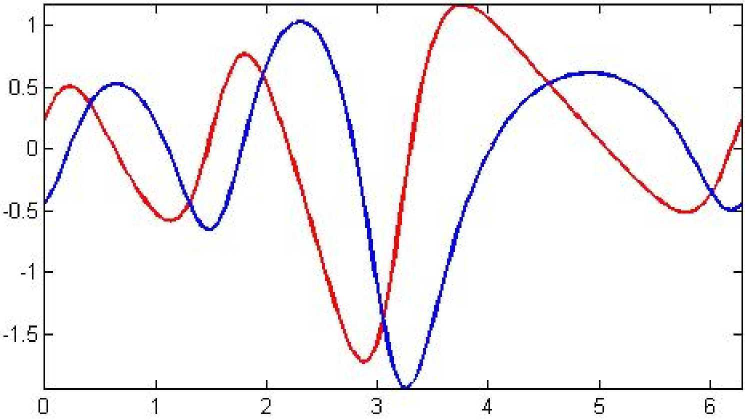
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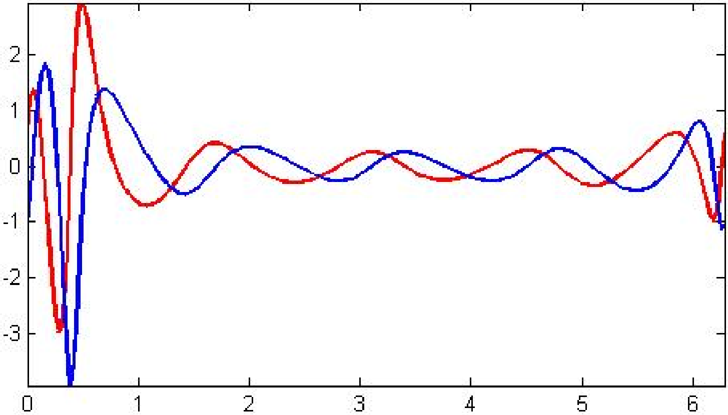

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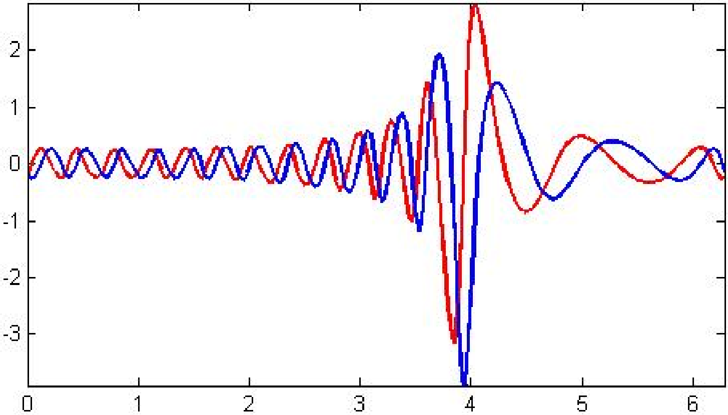
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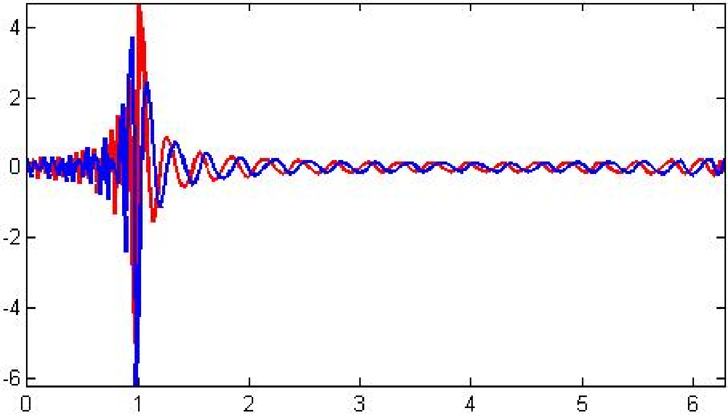
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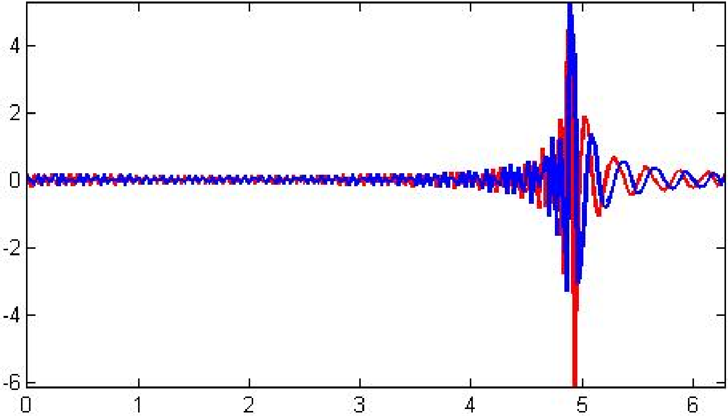

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Multiresolution analysis of $\mathcal{H}^2(\mathbb{T})$

- $$V_n = span\{\psi_{k,\ell},\ \ell = 0, 1, ..., 2^{2k} - 1,\ k = \overline{0, n}\}.$$
- The Malmquist -Takenaka system corresponding to the set A is a complete orthonormal system of holomorphic functions in $H^2(T)$, consequently the density condition is satisfied:
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The projection operator to the n-th resolution level

- $$V_{n+1} = V_n \bigoplus W_n.$$
- For $f \in H^2(T)$ let consider:

$$P_n f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} \langle f, \psi_{k,\ell} \rangle \psi_{k,\ell}(z)$$

,

$$\psi_{m,\ell}(z) = \frac{\sqrt{1-r_m^2}}{1-\overline{z_{m\ell}}z} \prod_{k=0}^{m-1} \prod_{j=0}^{2^{2k}-1} \frac{z-z_{kj}}{1-\overline{z_{kj}}z} \prod_{j'=0}^{\ell-1} \frac{z-z_{mj'}}{1-\overline{z_{mj'}}z}.$$

Theorem

For $f \in H^2(T)$ *the projection operator* $P_n f$ *is an interpolation operator in the points* $z_{mj} = r_m e^{i\frac{2\pi j}{2^{2m}}}$, $(j = 0, ..., 2^{2m} - 1, \quad m = 0, ..., n)$ *for the analytic continuation of* f *in the unit disc,*

$$\|f - P_n f\| \to 0, \quad n \to \infty,$$

uniform convergence for the analytic continuation of f *inside the unit disc on every compact subset. For every* $f \in H^2(D)$

$$\|P_n f(z) - f\| = \inf_{f_n \in V_n} \|f_n - f\|,$$

Reconstruction algorithm

- In what follows we propose a computational scheme in the wavelet base $\{\psi_{k,\ell},\ \ell = 0, 1, \ldots, 2^{2k} - 1,\ k = 0, \ldots, n\}$.
- The projection of $f \in H^2(T)$ onto V_{n+1} can be written in the following way:

$$Q_n f(z) := \sum_{\ell=0}^{2^{2(n+1)}-1} \langle f, \psi_{n+1,\ell} \rangle \psi_{n+1,\ell}(z),$$

$$P_{n+1} f = P_n f + Q_n f,\ \ Q_n f(z_{k\ell}) = 0, k = \overline{1, n},\ \ \ell = \overline{0, 2^{2n} - 1}.$$

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- The set of coefficients of the best approximant $P_n f$: $(\{b_{k\ell} = \langle f, \psi_{k,\ell} \rangle, \ \ell = 0.1, ... 2^{2k} - 1 \ \ k = 0, 1, ..., n\})$ is the (discrete) hyperbolic wavelet transform of the function f.
- The coefficients of the projection operator $P_n f$ can be computed if we know the values of the functions on $\bigcup_{k=0}^{n} A_k$.

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Reconstruction algorithm

For $f \in H^2(T)$

$$P_n f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} \langle f, \psi_{k,\ell} \rangle \psi_{k,\ell}(z),$$

$$\langle f, \psi_{k,\ell} \rangle = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{2^{2k'}-1} \overline{c_{k',\ell'}} f(z_{k',\ell'}) + \sum_{j=0}^{\ell} \overline{c_{k,j}} f(z_{k,j}),$$

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Summary

Measuring the values of the function f in the points of the set $A = \bigcup_{k=0}^{n} A_k \subset D$ we can write the the projection operator at the n-th resolution level which is convergent in $H^2(T)$ norm on the unit circle to f, is the best approximant interpolation operator on the set the $\bigcup_{k=0}^{n} A_k$ inside the unit circle for the analytic continuation of f and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc. Complex coloring visualization of hyperbolic wavelets see homepage of Levente Lócsi: http://locsi.web.elte.hu/.

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