

# A dominated ergodic theorem for some bilinear averages (JMAA, 2010)

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  - The discrete bilinear Hardy-Littlewood maximal operator
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# The bilinear problem

$T : L^{p_i}(\mu) \rightarrow L^{p_i}(\mu), i = 1, 2, (p_i > 1),$

$T, T^{-1} \geq 0 (f \geq 0 \Rightarrow Tf, T^{-1}f \geq 0).$

$$\mathcal{A}_n f(x) = \frac{1}{2n+1} \sum_{i=-n}^n T^i f(x)$$

Assume that we know that the linear averages converge: there exists

$$\lim_{n \rightarrow \infty} \mathcal{A}_n f \quad \text{in } L^{p_i}(\mu), \text{ for all } f \in L^{p_i}(\mu), i = 1, 2.$$

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## Bilinear averages

Let  $\mathcal{A}_n(f_1, f_2) = (\mathcal{A}_n f_1)(\mathcal{A}_n f_2)$ . Then

$\exists \lim_{n \rightarrow \infty} \mathcal{A}_n(f_1, f_2) \text{ in } L^p(\mu), \text{ for all } f_1 \in L^{p_1}(\mu) \text{ and all } f_2 \in L^{p_2}(\mu),$

where  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ .

### Questions:

- May we have the convergence of  $\mathcal{A}_n(f_1, f_2)$  without the convergence of the linear averages?
- May we characterize the convergence of  $\mathcal{A}_n(f_1, f_2)$  in  $L^p(\mu)$ ,  $f_1 \in L^{p_1}(\mu)$ ,  $f_2 \in L^{p_2}(\mu)$ ?
- Is the convergence equivalent to the uniform boundedness of  $\mathcal{A}_n(f_1, f_2)$ ?
- What can we say about the almost everywhere convergence?



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# The setting

- $(X, \mathcal{F}, \mu)$ ,  $\sigma$ -finite measure space.
- $T : L^p(\mu) \rightarrow L^p(\mu)$  invertible linear operator,  $1 < p < \infty$ ,  
 $T, T^{-1} \geq 0$ :
  - $Tf(x) = g(x)f(\tau x)$ ,  $g > 0$
  - $\tau : X \rightarrow X$ , measurable, invertible, non singular:

$$\mu(E) = 0 \Leftrightarrow \mu(\tau E) = 0 \Leftrightarrow \mu(\tau^j E) = 0$$

- $T^i f(x) = g_i(x)f(\tau^i x)$ ; for  $i \geq 1$ ,  $g_i(x) = \prod_{j=1}^{i-1} g(\tau^j x)$ .
- $\nu_i(E) = \mu(\tau^i E)$ ,  $J_i(x) = \frac{d\nu_i}{d\mu}(x)$ .
- If  $H_i = g_i^{-p} J_i$  then

$$\int_X |T^i f(x)|^p H_i(x) d\mu(x) = \int_X |f(x)|^p d\mu(x)$$

- $\mathcal{A}_n f(x) = \frac{1}{2n+1} \sum_{i=-n}^n T^i f(x)$ ,  $\mathcal{M}f(x) = \sup_{n \geq 0} |\mathcal{A}_n f|(x)$ .

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# Special case: the discrete Hardy-Littlewood maximal operator

$\tau : \mathbb{Z} \rightarrow \mathbb{Z}, \tau(x) = x + 1, g(x) = 1, a : \mathbb{Z} \rightarrow \mathbb{R}, a = \{a(j)\},$

$$Ta(j) = a(j+1)$$

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**Weights,  $1 < p < \infty$ ,  $w : \mathbb{Z} \rightarrow [0, \infty)$  (Muckenhoupt and Hunt, Muckenhoupt and Wheeden)**

The following are equivalent.

- $\sum_{j=-\infty}^{\infty} |ma(j)|^p w(j) \leq C \sum_{j=-\infty}^{\infty} |a(j)|^p w(j)$
- $w \in A_p: \left( \sum_{j=m}^n w(j) \right)^{\frac{1}{p}} \left( \sum_{j=m}^n w^{1-p'}(j) \right)^{\frac{1}{p'}} \leq C(n-m+1),$   
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# The result in the linear case

## Theorem (de la Torre, M-R, 1985/1988)

For  $1 < p < \infty$ , the following statements are equivalent.

- (a)  $\int_X |\mathcal{M}f|^p d\mu \leq C \int_X |f|^p d\mu.$
- (b)  $\sup_n \|\mathcal{A}_n f\|_p \leq C \|f\|_p.$  ← CNecesaria  $L^{2p}$  ← CNecesariaw ← CNecesariaw<sub>1</sub>
- (c)  $\mathcal{A}_n f$  converges in  $L^p(\mu)$  for all  $f \in L^p(\mu).$
- (d) For a. e.  $x \in X$ , the function  $i \rightarrow H_i(x)$  satisfies Muckenhoupt  $A_p$  condition (on the integers), with a constant independent of  $x.$
- (e) These equivalent conditions imply that  $\mathcal{A}_n f$  converges a.e. for all  $f \in L^p(\mu).$

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- (a) $\Rightarrow$ (b), obvious
- (b) $\Rightarrow$ (d), using Rubio de Francia algorithm of factorization of weights.
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# The bilinear case

$$\mathcal{A}_n(f_1, f_2)(x) = \frac{1}{(2n+1)^2} \left( \sum_{i=-n}^n T^i f_1(x) \right) \left( \sum_{i=-n}^n T^i f_2(x) \right)$$

## Questions:

- May we characterize the convergence of  $\mathcal{A}_n(f_1, f_2)$  in  $L^p(\mu)$ ,  $f_1 \in L^{p_1}(\mu)$ ,  $f_2 \in L^{p_2}(\mu)$ ? ( $p_i > 1$ ,  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ )
- Is it equivalent to the uniform boundedness of the bilinear averages  $\mathcal{A}_n(f_1, f_2)$ ?
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$$\mathcal{M}(f_1, f_2)(x) = \sup_n \mathcal{A}_n(|f_1|, |f_2|)(x)?$$

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## 3 The bilinear case

- The discrete bilinear Hardy-Littlewood maximal operator
- The bilinear ergodic maximal operator
- Open problems

# The discrete bilinear Hardy-Littlewood maximal operator

Particular case:  $X = \mathbb{Z}$ ,  $\tau x = x + 1$ ,  $g(x) = 1$ .

## Definition: The discrete bilinear Hardy-Littlewood maximal operator

Let  $a, b : \mathbb{Z} \rightarrow \mathbb{R}$

$$A_n(a, b)(j) = \left( \frac{1}{2n+1} \sum_{i=-n}^n a(j+i) \right) \left( \frac{1}{2n+1} \sum_{i=-n}^n b(j+i) \right).$$

$$m(a, b)(j) = \sup_{n \geq 0} |A_n(|a|, |b|)(j)|.$$

# Boundedness of the bilinear Hardy-Littlewood maximal operator

## Theorem[2009,Advances in Math.] [LOPTT].

Let  $w, u, v : \mathbb{Z} \rightarrow [0, \infty)$ ,  $w = u^{p/p_1} v^{p/p_2}$ ,  $1/p_1 + 1/p_2 = 1/p$ ,  $1 < p_i < \infty$ . The following assertions are equivalent.

- a)  $\left( \sum_{j=-\infty}^{\infty} |m(a, b)(j)|^p w(j) \right)^{\frac{1}{p}} \leq C \left( \sum_{j=-\infty}^{\infty} |a(j)|^{p_1} u(j) \right)^{\frac{1}{p_1}} \left( \sum_{j=-\infty}^{\infty} |b(j)|^{p_2} v(j) \right)^{\frac{1}{p_2}}$
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# The ergodic bilinear problem

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## The difficult implication

$$\sup_n \|\mathcal{A}_n(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \Rightarrow \|\mathcal{M}(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

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## The difficult implication

$$\sup_n \|\mathcal{A}_n(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \Rightarrow \|\mathcal{M}(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

## Boundedness of the bilinear ergodic maximal operator: first step

$$\sup_n \|\mathcal{A}_n(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \Rightarrow \sup_n \|\mathcal{A}_n(f)\|_{2p} \leq C \|f\|_{2p}$$

$$\begin{aligned}\mathcal{A}_n(f) &= \mathcal{A}_n(f^{p/p_1} f^{p/p_2}) \\ &\leq \left( \mathcal{A}_n(f^{2p/p_1}) \right)^{1/2} \left( \mathcal{A}_n(f^{2p/p_2}) \right)^{1/2}\end{aligned}$$

$$\begin{aligned}\int |\mathcal{A}_n(f)|^{2p} &\leq \int \left( \mathcal{A}_n(f^{2p/p_1}) \right)^p \left( \mathcal{A}_n(f^{2p/p_2}) \right)^p \\ &= \int \left( \mathcal{A}_n(f^{2p/p_1}, f^{2p/p_2}) \right)^p \\ &\leq C \left( \int f^{(2p/p_1)p_1} \right)^{p/p_1} \left( \int f^{(2p/p_2)p_2} \right)^{p/p_2} = C \int f^{2p}\end{aligned}$$

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## Consequence

For all  $j$ , if  $H_j = g_j^{-2p} J_j$

$$\int_X |T^j f(x)|^{2p} H_j(x) d\mu(x) = \int_X |f(x)|^{2p} d\mu(x)$$

► Caso Lineal

- For a.e.  $x \in X$ , the function  $j \rightarrow H_j(x)$  satisfies  $A_{2p}$ , with a constant independent of  $x$ .

◀ Transferencia

$$\int_X |T^j f(x)|^p H_j(x) g_j^p(x) d\mu(x) = \int_X |f(x)|^p d\mu(x)$$

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## Other weights

$$w(x, j) = H_j(x), \quad u(x, j) = \frac{H_j(x)}{(g_j(x))^{p_1 - 2p}}, \quad v(x, j) = \frac{H_j(x)}{(g_j(x))^{p_2 - 2p}}.$$

$$\int_X |T^j f(x)|^{2p} w(x, j) d\mu(x) = \int_X |f(x)|^{2p} d\mu(x)$$

Transferencia

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$$\sup_n \|\mathcal{A}_n(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}$$



$$\|\mathcal{A}_n^T(f)\|_{2p} \leq C \|f\|_{2p}, \quad \|\mathcal{A}_n^R(f)\|_{2p'_1} \leq C \|f\|_{2p'_1}, \quad \|\mathcal{A}_n^R(f)\|_{2p'_2} \leq C \|f\|_{2p'_2}$$

$$\begin{aligned} & \left( \int_X |T^j f(x)|^{2p} w(x, j) d\mu(x) = \int_X |f(x)|^{2p} d\mu(x) \right) \\ & \left( \int_X |R^j f(x)|^{2p'_1} u^{1-p'_1}(x, j) d\mu(x) = \int_X |f(x)|^{2p'_1} d\mu(x) \right) \\ & \left( \int_X |R^j f(x)|^{2p'_2} v^{1-p'_2}(x, j) d\mu(x) = \int_X |f(x)|^{2p'_2} d\mu(x) \right) \end{aligned}$$



For a.e.  $x$  and with a constant independent of  $x$

- The function  $j \rightarrow H_j(x) = w(x, j) \in A_{2p}$ .
- The function  $j \rightarrow u^{1-p'_1}(x, j) \in A_{2p'_1}$ . (It is proved using duality:  $p \geq 1$ )
- The function  $j \rightarrow v^{1-p'_2}(x, j) \in A_{2p'_2}$ . (In a symmetric way)

## Transference Argument: Notations

- $\mathcal{M}_\eta(f_1, f_2)(x) = \sup_{0 < n \leq \eta} \mathcal{A}_n(f_1, f_2)(x)$ . ( $f_i \geq 0$ )
- $f^x(j) = T^j f(x)$ ,

## Transference Argument

For all  $R > 0$

$$\int_X (\mathcal{M}_\eta(f_1, f_2)(x))^p d\mu(x) = \int_X (T^j \mathcal{M}_\eta(f_1, f_2)(x))^p H_j(x) g_j^p(x) d\mu$$

» Jacobiano

$$= \frac{1}{R+1} \int_X \sum_{j=0}^R (T^j \mathcal{M}_\eta(f_1, f_2)(x))^p H_j(x) g_j^p(x) d\mu(x).$$

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$$T \geq 0 \Rightarrow T^j(\mathcal{M}_\eta(f_1, f_2))(x) \leq m(f_1^x \chi_{[-\eta, R+\eta]}, f_2^x \chi_{[-\eta, R+\eta]})(j) \frac{1}{g_j(x)}$$

## Transference Argument

For all  $R > 0$

$$\begin{aligned} & \int_X (\mathcal{M}_\eta(f_1, f_2)(x))^p d\mu(x) \\ &= \frac{1}{R+1} \int_X \sum_{j=0}^R |T^j \mathcal{M}_\eta(f_1, f_2)(x)|^p H_j(x) (g_j(x))^p d\mu(x) \\ &\leq \frac{1}{R+1} \int_X \sum_{j=0}^R |m(f_1^x \chi_{[-\eta, R+\eta]}, f_2^x \chi_{[-\eta, R+\eta]})(j)|^p H_j(x) d\mu(x) \\ &\leq \frac{1}{R+1} \int_X \sum_{j=-\infty}^{\infty} |m(f_1^x \chi_{[-\eta, R+\eta]}, f_2^x \chi_{[-\eta, R+\eta]})(j)|^p w(x, j) d\mu(x) \end{aligned}$$

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## Transference Argument

For all  $R > 0$

$$\begin{aligned} & \int_X (\mathcal{M}_\eta(f_1, f_2)(x))^p d\mu(x) \\ &= \frac{1}{R+1} \int_X \sum_{j=0}^R |T^j \mathcal{M}_\eta(f_1, f_2)(x)|^p H_j(x) (g_j(x))^p d\mu(x) \\ &\leq \frac{1}{R+1} \int_X \sum_{j=0}^R |m(f_1^x \chi_{[-\eta, R+\eta]}, f_2^x \chi_{[-\eta, R+\eta]})(j)|^p H_j(x) d\mu(x) \\ &\leq \frac{1}{R+1} \int_X \sum_{j=-\infty}^{\infty} |m(f_1^x \chi_{[-\eta, R+\eta]}, f_2^x \chi_{[-\eta, R+\eta]})(j)|^p w(x, j) d\mu(x) \end{aligned}$$

$$T \geq 0 \Rightarrow T^j(\mathcal{M}_\eta(f_1, f_2))(x) \leq m(f_1^x \chi_{[-\eta, R+\eta]}, f_2^x \chi_{[-\eta, R+\eta]})(j) \frac{1}{g_j(x)}$$

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$$w(x, j) = H_j(x), \quad u(x, j) = \frac{H_j(x)}{(g_j(x))^{p_1 - 2p}}, \quad v(x, j) = \frac{H_j(x)}{(g_j(x))^{p_2 - 2p}}.$$

$w(x, \cdot) \in A_{2p}$ ,  $u^{1-p'_1}(x, \cdot) \in A_{2p'_1}$ ,  $v^{1-p'_2}(x, \cdot) \in A_{2p'_2}$  (with a constant independent of  $x$ ).

$$w^{1/p} = u^{1/p_1} v^{1/p_2}$$

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► [LOPTT]

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$$\times \left( \sum_{j=-\eta}^{R+\eta} \int_X |T^j f_2(x)|^{p_2} v(x, j) d\mu(x) \right)^{p/p_2}$$

Jacobiano

$$= \frac{C}{R+1} \left( (R+2\eta+1) \int_X |f_1|^{p_1} d\mu \right)^{p/p_1} \left( (R+2\eta+1) \int_X |f_2|^{p_2} d\mu \right)^{p/p_2}$$

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$$\begin{aligned} & \int_X (\mathcal{M}_\eta(f_1, f_2)(x))^p d\mu(x) \\ & \leq C \frac{R + 2\eta + 1}{R + 1} \left( \int_X |f_1|^{p_1} d\mu(x) \right)^{p/p_1} \left( \int_X |f_2|^{p_2} d\mu(x) \right)^{p/p_2} \rightarrow \\ & \rightarrow C \left( \int_X |f_1|^{p_1} d\mu(x) \right)^{p/p_1} \left( \int_X |f_2|^{p_2} d\mu(x) \right)^{p/p_2} \end{aligned}$$

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## WE HAVE PROVED

$$\sup_n \|\mathcal{A}_n(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}$$



$$w(x, j) = H_j(x), u(x, j) = \frac{H_j(x)}{(g_j(x))^{p_1-2p}}, v(x, j) = \frac{H_j(x)}{(g_j(x))^{p_2-2p}}.$$

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$$\|\mathcal{M}(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}$$

# Main Result

## Theorem

If  $1/p_1 + 1/p_2 = 1/p$ ,  $1 < p_i < \infty$ ,  $1 \leq p < \infty$ , the following assertions are equivalent.

- (a)  $\mathcal{A}_n(f_1, f_2)$  converges in  $L^p(\mu)$ ,  $f_1 \in L^{p_1}(\mu)$  and  $f_2 \in L^{p_2}(\mu)$ .
- (b)  $\sup_{n \geq 0} \|\mathcal{A}_n(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}$   $f_1 \in L^{p_1}(\mu)$  and  $f_2 \in L^{p_2}(\mu)$ .
- (c)  $\|\mathcal{M}(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}$ .  $f_1 \in L^{p_1}(\mu)$  and  $f_2 \in L^{p_2}(\mu)$ .
- (d) For a.e.  $x$ ,  $w(x, \cdot) \in A_{2p}$ ,  $u^{1-p'_1}(x, \cdot) \in A_{2p'_1}$ ,  $v^{1-p'_2}(x, \cdot) \in A_{2p'_2}$  (with a constant independent of  $x$ )

Each one of the above conditions implies that

- (e)  $\mathcal{A}_n(f_1, f_2)$  converges a.e for  $f_1 \in L^{p_1}(\mu)$  and  $f_2 \in L^{p_2}(\mu)$ .

# Some consequence

## General Examples

Using the characterization, it is possible to give very general examples such that

- (a)  $\mathcal{A}_n(f_1, f_2)$  converges in  $L^p(\mu)$  for all  $f_1 \in L^{p_1}(\mu)$  and all  $f_2 \in L^{p_2}(\mu)$ .
- (b)  $\mathcal{A}_n(f_1)$  does not converge in  $L^{p_1}(\mu)$  for some  $f_1 \in L^{p_1}(\mu)$ .
- (c)  $\mathcal{A}_n(f_2)$  converges in  $L^{p_2}(\mu)$  for all  $f_2 \in L^{p_2}(\mu)$ .

## Theorem

Let  $(X, \mathcal{F}, \nu)$  be a non atomic finite measure space. Let  $\tau : X \rightarrow X$  be an invertible ergodic measure preserving transformation. Let  $1 < p, p_1, p_2 < \infty$ ,  $1/p_1 + 1/p_2 = 1/p$ ,  $p_1 \neq p_2$ . Then there exist a measure  $\mu$  equivalent to  $\nu$  ( $\mu(E) = 0 \Leftrightarrow \nu(E) = 0$ ) and a positive measurable function  $g$  such that the operator

$$Tf(x) = g(x)f(\tau x).$$

has the following properties:

- (a) The bilinear averages  $A_n(f_1, f_2)$  associated to  $T$  are uniformly bounded from  $L^{p_1}(\mu) \times L^{p_2}(\mu)$  into  $L^p(\mu)$
- (b) The linear averages  $A_n$  associated to  $T$  are not uniformly bounded in  $L^{p_1}(\mu)$
- (c) The linear averages  $A_n$  associated to  $T$  are uniformly bounded in  $L^{p_2}(\mu)$ .

1 Introduction

2 The result for the linear case

## 3 The bilinear case

- The discrete bilinear Hardy-Littlewood maximal operator
- The bilinear ergodic maximal operator
- Open problems

## Open problems

$$\bullet \quad 1/p_1 + 1/p_2 = 1/p, \quad 1 < p_i < \infty, \quad 0 < p < 1$$

$$\bullet \quad A_n(f_1, f_2)(x) = \frac{1}{n+1} \sum_{j=0}^n T^j f_1(x) \frac{1}{n+1} \sum_{j=0}^n T^j f_2(x).$$

$$\bullet \quad A_n(a, b)(j) = \frac{1}{n+1} \sum_{l=0}^n a(j+l) \frac{1}{n+1} \sum_{l=0}^n b(j+l).$$

# Open problems

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- $A_n(a, b)(j) = \frac{1}{n+1} \sum_{i=0}^n a(j+i) \frac{1}{n+1} \sum_{i=0}^n b(j+i).$