EXTREME LIMITING BEHAVIOR OF THE PARTIAL SUMS OF "SMOOTH" FUNCTIONS: THE DISK ALGEBRA

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1. Introduction

The problem

$$A(D) \quad = \text{disk algebra}.$$

$$\text{If } f \in A(D) \text{ then } S_n(f,0) \to f \text{ a.e. on } \mathbb{T}$$

(it follows by Carleson's theorem).
Question: What is the limiting behavior of the partial sums of f on "small" subsets of the unit circle?

Definition 1.1. Let $K \subset \mathbb{C} \setminus D$ be a countable set. We say that the partial sums of the Taylor development of a function $f \in H(D)$ with center 0 enjoy a pointwise universality property on K, if for every function $g : K \to \mathbb{C}$ there exists a subsequence (λ_n) of positive integers such that

$$\lim_{n\to+\infty} S_{\lambda_n}(f,0) = g \quad \text{pointwise on } K.$$

We denote the class of such functions by $U_p(D,K,0)$.

Definition 1.2. Let $K \subset \mathbb{C} \setminus D$ be a compact set. We say that the partial sums of the Taylor development of a function $f \in H(D)$ with center 0 enjoy a uniform universality property on K, if for every function $g \in A(K)$ there exists a subsequence (λ_n) of positive integers such that

$$\lim_{n\to+\infty} S_{\lambda_n}(f,0) = g \quad \text{uniformly on } K.$$

We denote the class of such functions by $U(D,K,0)$.

2. The results

Theorem 2.1. *Let $E \subset \mathbb{T}$ be a countable set. Then quasi all $f \in A(D)$ enjoy the property that for every function $h : E \to \mathbb{C}$ there is a subsequence of $(S_n(f,0))$ converging pointwise to h on E; equivalently, the set $U_p(D,E,0) \cap A(D)$ is G_δ and dense in $A(D)$. In particular, $U_p(D,E,0) \cap A(D) \neq \emptyset$.*

Denoting by $\mathcal{K}(\mathbb{T})$ the complete metric space of all compact, non-empty subsets of $\mathbb{T}$ equipped with the Hausdorff metric, we obtain

Theorem 2.2. *Quasi all functions $f \in A(D)$ enjoy the property that $(S_n(f,0))$ is uniformly universal on quasi all sets $E \subset \mathcal{K}(\mathbb{T})$. In particular, there exists a compact set $K \subset \mathbb{T}$ which is perfect and thus uncountable such that $U(D,K,0) \cap A(D) \neq \emptyset$.*

By Theorem 2.2 there are functions f in $A(D)$ enjoying the property that their partial sums $(S_n(f,0))$ are uniformly universal on sets $E \subset \mathcal{K}(\mathbb{T})$, where E is infinite. On the other hand, as the next proposition shows, there are certain sets $C \subset \mathbb{T}$ with $|C| = |\mathbb{N}|$ such that whenever the partial sums of a function $f \in H(D)$ are uniformly universal on C then $f \notin A(D)$. These sets arise as an application of Rogosinski's formula. In particular, a sequence $(e^{i\theta_n})$, $\theta_n \in \mathbb{R}$, of the unit circle is such a set provided that $e^{i\theta_n} \to 1$ "rather slowly"; for instance the choice $\theta_n = \pi/n$ suffices.

Proposition 2.3. *There are compact sets $C \subset \mathbb{T}$ such that*
(i) C is countable,
(ii) $U_p(D,C,0) \cap A(D) \neq \emptyset$,
(iii) $U(D,C,0) \cap A(D) = \emptyset$.

3. Sketch of Proofs

Lemma 3.1. *Let $\Lambda \subset \mathbb{N}$. Then the set of functions $f \in A(D)$ such that the sequence $(S_n(f,0)(1))_{n\in\Lambda}$ is unbounded is G_δ and dense in $A(D)$.*

Proof. Consider the so called Fejer polynomials

$$P_n(z) = \left(\frac{1}{n} + \frac{z}{n-1} + \ldots + \frac{z^{n-1}}{1}\right) - \left(\frac{z^n}{1} + \frac{z^{n+1}}{2} + \ldots + \frac{z^{2n-1}}{n}\right),$$

$n = 1, 2, \ldots$. Fejer showed that there is a positive number $M > 0$ such that $\|P_n\| = \sup_{z\in\mathbb{T}} |P_n(z)| \leq M$ for every $n \in \mathbb{N}$, i.e. the polynomials P_n are uniformly bounded on $\mathbb{T}$; On the other hand we have

$$S_n(P_n, 0)(1) = \frac{1}{n} + \frac{1}{n-1} + \ldots + 2 > \log n - 1$$

for every $n \in \mathbb{N}$. The maps $L_n : A(D) \to \mathbb{C}$, $L_n(f) = S_n(f,0)(1)$, $n \in \lambda$, $f \in A(D)$, are continuous linear functionals on $A(D)$. In view of the above properties of Fejer polynomials, these functionals are not uniformly bounded and consequently by the uniform boundedness theorem we conclude the existence of a function $g \in A(D)$ such that the sequence $(S_n(g,0)(1))_{n\in\Lambda}$ is unbounded. To show that such functions form a G_δ and dense set in $A(D)$ either one can apply the Banach-Steihaus theorem or one may use a more straightforward argument based on the facts that the polynomials are dense in $A(D)$ and that for every polynomial the sequence $(S_n(g+p,0)(1))_{n\in\Lambda}$ is unbounded. $\square$

Lemma 3.2. *Let $f \in A(D)$, $w \in \mathbb{T}$ and define $g(z) := (z-w)f(z)$, $z \in D$. Then we have*

$$S_n(g,0)(w) \to 0 \quad \text{as } n \to +\infty.$$

Proof. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in D$. Straightforward calculations show that for every non-negative integer n,

$$S_n(g,0)(z) = (z-w)S_n(f,0)(z) - a_n z^{n+1} \quad \text{for every } z \in \mathbb{C}.$$

Applying the above formula for $z = w$ we get

$$S_n(g,0)(w) = -a_n w^{n+1},\ n = 0,1,2,\ldots$$

and since $a_n \to 0$ (recall that the Taylor coefficients of a function belonging to the disk algebra, tend to zero) we reach our conclusion. $\square$

Lemma 3.3. *Let E be a finite subset of $\mathbb{T}$. Then $U_p(D,E,0) \cap A(D)$ is G_δ and dense in $A(D)$.*

Proof. **Step 1**. Let Λ be an infinite set of positive integers. We shall prove that the set

$$\{f \in A(D) : \text{the set } \{S_n(f,0)(1) : n \in \Lambda\} \text{ is dense in } \mathbb{C}\}$$

is G_δ and dense in $A(D)$.

In order to prove the last assertion we first show that given any $g \in A(D)$, $c \in \mathbb{C}$, $\epsilon > 0$ there exist $f \in A(D)$ and $n \in \Lambda$ such that

$$\|f-g\| < \epsilon \quad \text{and} \quad S_n(f,0)(1) = c.$$

Mergelyan's theorem implies the existence of a polynomial p with $\|g-p\| < \epsilon/2$. By Lemma 3.1 the set of $f \in A(D)$ such that the sequence $(S_n(f,0)(1))_{n\in\Lambda}$ is unbounded is G_δ and dense in $A(D)$. Hence, there exist $h \in A(D)$ and $n \in \Lambda$ with $n \geq \deg(p)$ such that

$$\|h\| < \frac{\epsilon}{2} \quad \text{and} \quad |S_n(h,0)(1)| > |c - p(1)|.$$

Then the function

$$\Phi := \frac{c - p(1)}{S_n(h,0)(1)} h$$

belongs to $A(D)$ and satisfies

$$S_n(\Phi,0)(1) = c - p(1) \quad \text{and} \quad \|\Phi\| \leq \|h\| < \frac{\epsilon}{2}.$$

Thus for $f := \Phi + p \in A(D)$ and since $n \geq \deg(p)$ we obtain

$$\|f-g\| < \epsilon \quad \text{and} \quad S_n(f,0)(1) = S_n(\Phi,0)(1) + p(1) = c.$$

According to the Universality Criterion applied to the sequence $T_n : A(D) \to \mathbb{C}$, $n \in \Lambda$ with $T_n f = S_n(f,0)(1)$ for $f \in A(D)$, $n \in \Lambda$ the set

$$\{f \in A(D) : \text{the set } \{S_n(f,0)(1) : n \in \Lambda\} \text{ is dense in } \mathbb{C}\}$$

is G_δ and dense in $A(D)$.

Step 2. We prove the assertion by induction on $N = |E|$. For $N = 1$ the result follows from Step 1, where without loss of generality we may suppose that $E = \{1\}$. Let now $E \subset \mathbb{T}$ with $|E| = N + 1$ and our inductive hypothesis is that for every subset of $\mathbb{T}$ with N points the assertion holds true. Without loss of generality we may assume that $1 \in E$. Then we can write $E = F \cup \{1\}$ where $|F| = N$. The universality criterion, applied to

$$T_n : A(D) \to \mathbb{C}^E, T_n f := S_n(f,0)|_E, n \in \mathbb{N}, f \in A(D),$$

shows that it suffices to guarantee that for every $g \in A(D)$, every $\epsilon > 0$ and every function $h : E \to \mathbb{C}$ there exist $f \in A(D)$ and a positive integer n such that

$$\|f - g\| < \epsilon \quad \text{and} \quad \|S_n(f,0) - h\|_E < \epsilon.$$

By Mergelyan's theorem we may assume that g is a polynomial. Fix an entire function ϕ having the following interpolation properties:

$$\phi|_F = 1 \quad \text{and} \quad \phi(1) = 0$$

and set $M := \sup_{|z|\le 1} |\phi(z)|$. By induction hypothesis there exist $u \in A(D)$ and $\Lambda \subset \mathbb{N}$ with $|\Lambda| = \infty$ such that

$$\|u\| < \frac{\epsilon}{2M} \quad \text{and} \quad |S_n(u,0)(z) - (h(z) - g(z))| < \frac{\epsilon}{3}$$

for every $z \in F$ and every $n \in \Lambda$. By the Step 1 there exist $v \in A(D)$ and $\Lambda' \subset \Lambda$, $|\Lambda'| = \infty$ such that

$$\|v\| < \frac{\epsilon}{2(M+1)} \quad \text{and} \quad |S_n(v,0)(1) - (h(1) - g(1))| < \frac{\epsilon}{3}$$

for every $n \in \Lambda'$. Let now $w \in F$. Since $\phi(w) = 1$ there exists an entire function ψ such that $\phi(z) = 1 + (z - w)\psi(z)$, $z \in \mathbb{C}$. Set $\Psi(z) := (z - w)\psi(z)u(z)$ for $z \in D$. Then $\psi u \in A(D)$ and by Lemma 3.2 we conclude that

$$|S_n(u,0)(w) - S_n(u\phi,0)(w)| = |S_n(\Psi,0)(w)| \to 0 \quad \text{as } n \to +\infty.$$

Since F is a finite set it follows that

$$\|S_n(u,0) - S_n(u\phi,0)\|_F \to 0 \quad \text{as } n \to +\infty.$$

We follow a similar argument to control the quantity $|S_n(u\phi,0)(1)|$ for large n. Indeed, the function ϕ vanishes at 1, so there exists an entire function α such that $\phi(z) = (z - 1)\alpha(z)$, $z \in \mathbb{C}$. Set $A(z) :=$

$(z-1)u(z)\alpha(z)$, $z \in D$ and observe that $u\alpha \in A(D)$. Lemma 3.2 implies that

$$|S_n(u\phi, 0)(1)| = |S_n(A, 0)(1)| \to 0 \quad \text{as } n \to +\infty.$$

In a similar manner one shows that

$$|S_n(v\phi, 0)(1)| \to 0 \quad \text{as } n \to +\infty$$

and

$$|S_n(v, 0) - S_n(v\phi, 0)\|_F \to 0 \quad \text{as } n \to +\infty.$$

From the above we get

$$\|S_n(u\phi, 0) - S_n(u, 0)\|_F < \frac{\epsilon}{3}, \qquad |S_n(u\phi, 0)(1)| < \frac{\epsilon}{3}$$

and

$$|S_n(v\phi, 0)(1)| < \frac{\epsilon}{3}, \qquad \|S_n(v, 0) - S_n(v\phi, 0)\|_F < \frac{\epsilon}{3}$$

for n sufficiently large. Let us now define

$$f := u\phi + v(1-\phi) + g.$$

Then

$$\|f - g\|$$
$$\leq \|u\|\|\phi\| + \|v\|\|1-\phi\| < \frac{\epsilon}{2M}M + \frac{\epsilon}{2(M+1)}(M+1) = \epsilon$$

and for $n \in \Lambda'$ with $n \geq \deg g$ we have (since $S_n(g, 0) = g$)

$$\|S_n(f, 0) - h\|_F$$
$$\leq \|S_n(v\phi, 0) - S_n(v, 0)\|_F + \|S_n(u\phi, 0) + g - h\|_F$$
$$\leq \frac{\epsilon}{3} + \|S_n(u\phi, 0) - S_n(u, 0)\|_F + \|S_n(u, 0) - (h-g)\|_F < \epsilon$$

and similarly

$$|S_n(f, 0)(1) - h(1)|$$
$$\leq |S_n(u\phi, 0)(1)| + |S_n(v(1-\phi), 0)(1) + g(1) - h(1)| < \epsilon.$$

□

The proof of Proposition 2.3 relies heavily on a classical formula due to Rogosinski which connects the Cesaro summability of power series on a point of the unit circle, say 1, to the behavior of the partial sums near 1. Actually, we shall use the following variant of Rogosinki's formula which appears in a work of Melas and Nestoridis.

Lemma 3.4. *Let $(c_\nu)_{\nu\geq 0}$ be a sequence of complex numbers and $S_n(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ the associated Fourier series. Set $S_n = S_n(1)$. Suppose that the series $\sum_{\nu\geq 0} c_\nu$ is $(C,1)$ summable to $\sigma \in \mathbb{C}$. Let $\mathcal{D}$ be an infinite subset of $\mathbb{N}$ and for every $n \in \mathcal{D}$ let z_n be a complex number such that $\lim_{n\to+\infty, n\in\mathcal{D}} n(1-z_n) = u \neq 0$. Then*

$$\lim_{n\to+\infty, n\in\mathcal{D}} z_n^{-n}(S_n(z_n)-\sigma) - (S_n - \sigma) = 0.$$

Definition 3.5. A compact set $K \subset \mathbb{C} \setminus D$ is said to be non-admissible if

(i) $1 \in K$,

(ii) there exists a sequence (z_n) in K such that $n(1-z_n) \to u$ for some non-zero complex number u and $z_n^{-n} \to b$ for some complex number b with $b \neq 1$.

Proposition 3.6. *Let $K \subset \mathbb{C} \setminus D$ be a non-admissible compact set and let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in U(D, K, 0)$. Then the series $\sum_{n=0}^{\infty} a_n$ is not $(C,1)$ summable. In particular f does not belong to the disk algebra, i.e. $f \notin A(D)$.*

Proof. We argue by contradiction, so assume that the series $\sum_{n=0}^{\infty} a_n$ is $(C,1)$ summable to $\sigma \in \mathbb{C}$. Take a complex number a such that $a \neq \sigma$. By the universality property there exists a sequence of positive integers (p_m) such that

$$\sup_{z\in K} |S_{p_m}(f,0)(z) - a| \to 0 \quad \text{as } m \to +\infty.$$

Since K is non-admissible, see Definition 3.5, there exists a sequence (z_n) of complex numbers such that $z_n \in K$, $z_n^{-n} \to b$ with $b \neq 1$ and $n(1-z_n) \to u$ for some non-zero complex number u. By Lemma 3.4 we have

$$\lim_{m\to+\infty} \left(z_{p_m}^{-p_m}(S_{p_m}(f,0)(z_{p_m}) - \sigma) - (S_{p_m}(f,0)(1) - \sigma) \right) = 0$$

and by uniform convergence of $S_{p_m}(f,0)(z)$ on K, $S_{p_m}(f,0)(z_{p_m})$ converges to a. It now follows that

$$b(a-\sigma) - (a-\sigma) = 0$$

and since $b \neq 1$ we conclude that $a = \sigma$, which is a contradiction. □

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