Solution to a conjecture on the Hardy operator minus the identity and a new class of minimal rearrangement invariant spaces

Javier Soria

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Abstract

We give a positive answer to a conjecture of N. Kruglyak and E. Setterqvist about the norm of the Hardy operator minus the identity on decreasing functions.

– Joint work with S. Boza [JFA, 2011]

This study leads us to consider a new class of minimal rearrangement invariant spaces, for which we also establish some functional properties.

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Hardy operator minus the identity

The study of the Hardy operator

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on monotone functions has its origins in the works of Ariño-Muckenhoupt (TAMS, 1990) and Sawyer (Studia Math., 1990), extending the classical Hardy's inequalities:

If $\alpha > -1$, $p > \alpha + 1$, and $p \geq 1$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p x^\alpha\,dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty f(x)^p\,x^\alpha\,dx$$

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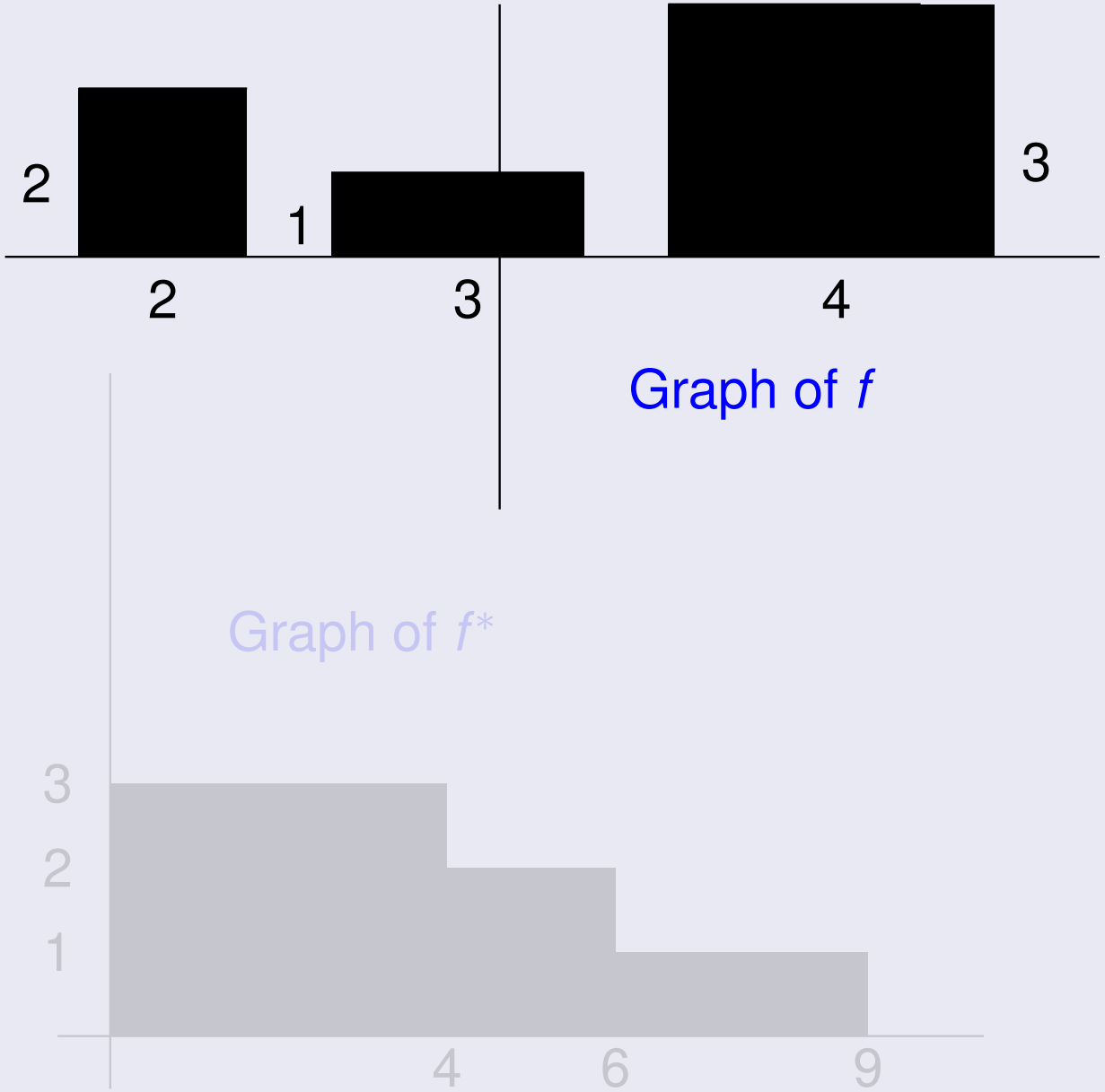
Decreasing rearrangement of f: f^*



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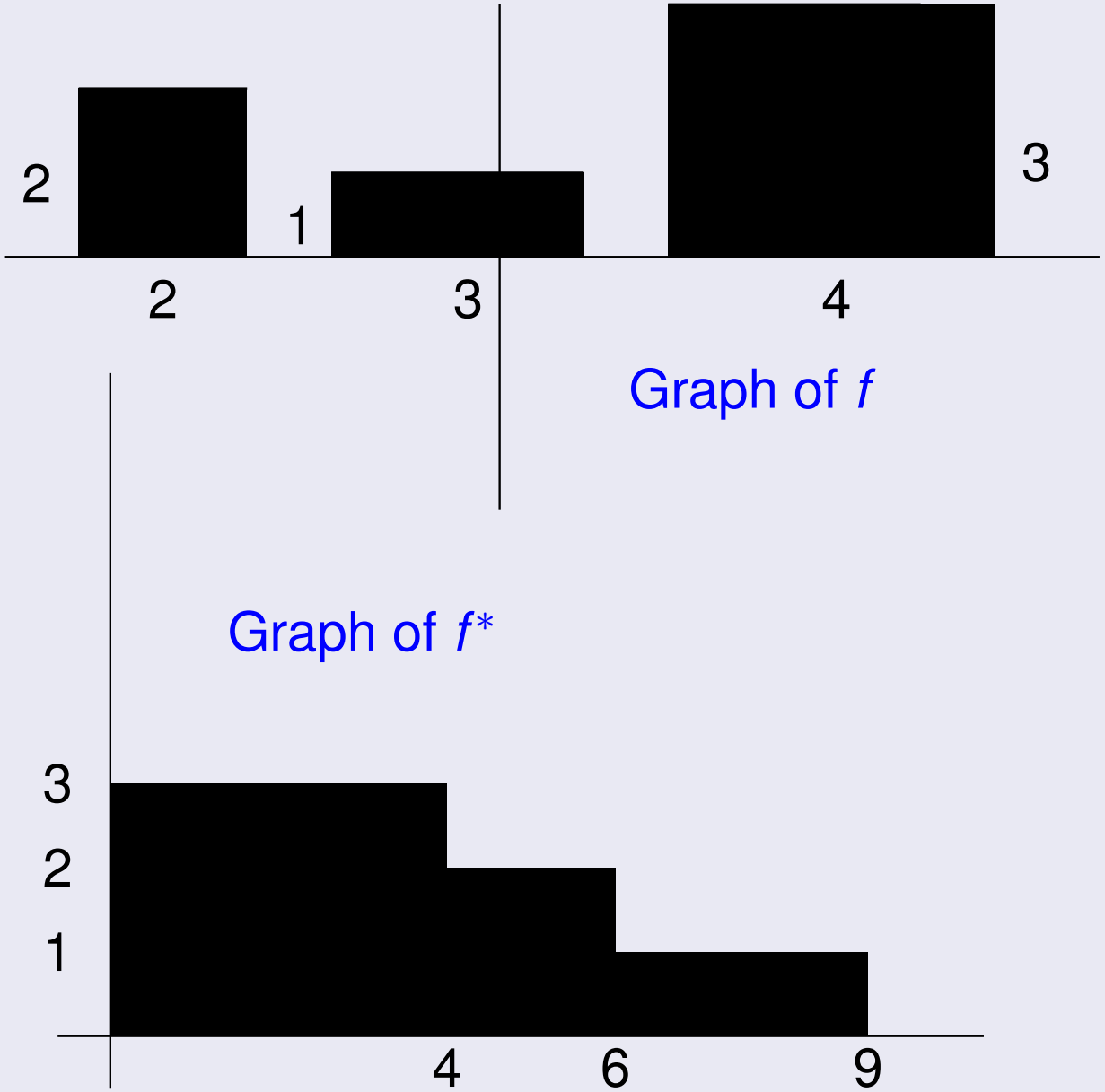

Motivation:

Boundedness of the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

on weighted Lorentz spaces:

$$\Lambda^p(w) := \left\{ f : \left(\int_0^\infty (f^*(t))^p\, w(t)\, dt \right)^{1/p} < \infty \right\}.$$

Since $(Mf)^* \approx S(f^*)$ (F. Riesz, Wiener, Herz), then

$$M : \Lambda^p(w) \to \Lambda^p(w),$$

if and only if (weighted Hardy's inequalities on monotone functions),

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p w(x)\, dx \le C \int_0^\infty f(x)^p\, w(x)\, dx, \quad f \downarrow$$

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For $p > 0$, we recall that a weight w is in the B_p-class if there exists a positive constant $C > 0$ such that, for every $r > 0$,

$$r^p \int_r^\infty \frac{w(x)}{x^p}\, dx \le C \int_0^r w(x)\, dx.$$

If $w \in B_p$ we denote by $\|w\|_{B_p}$ the best constant in the above inequality.

Theorem (Ariño-Muckenhoupt)

$S : L^p_{\mathrm{dec}}(w) \to L^p(w)$ if, and only if, $w \in B_p$.

Normability properties for $\Lambda^p(w)$ are also equivalent to $w \in B_p$, $p > 1$ (Sawyer), extending the well-known results of Lorentz (Ann. of Math., 1950).

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$$f^{**} - f^{*} = S(f^{*}) - \mathrm{Id}\,(f^{*}).$$

This is equivalent to considering the Hardy operator minus the Identity acting on decreasing functions, and measures the oscillation of the function f.

Finding good estimates for this operator has applications to, e.g., Sobolev-type embeddings and the Pólya–Szegö symmetrization principle.

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$$\|S - \mathrm{Id}\|_{L^p_{\mathrm{dec}}} = \frac{1}{(p-1)^{1/p}},$$

and conjectured that the result would be true for any $p \geq 2$.

Theorem (Boza – S. [JFA, 2011])

Let $p \geq 2$ and w be a weight in the B_p-class satisfying that

$$r^{p-1} \int_r^\infty \frac{w(x)}{x^p}\, dx,$$

is a decreasing function of $r > 0$. Then,

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Remarks

- $w \in B_p$ is a necessary condition for the boundedness of $S - \mathrm{Id} : L^p_{\mathrm{dec}}(w) \to L^p_{\mathrm{dec}}$.
- There are examples of weights w not satisfying that $r^{p-1} \int_r^\infty \frac{w(x)}{x^p}\, dx$ is a decreasing function for which the result is false.
- If $1 < p < 2$ there are also counterexamples.
- However, assuming only that $w \in B_1$,

$$\|S - \mathrm{Id}\|_{L^1_{\mathrm{dec}}(w)} = \|w\|_{B_1}.$$

- Kruglyak and Setterqvist's result corresponds to the unweighted case $w = 1$.

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Remark

It is yet an open problem to determine the norm of S on $L^p_{\mathrm{dec}}(w)$:

$$\|S\|_{L^p_{\mathrm{dec}}(w)} = \sup_{f\downarrow} \frac{\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f(x)\,dx\right)^p w(t)\,dt\right)^{1/p}}{\left(\int_0^\infty f^p(t)w(t)\,dt\right)^{1/p}}.$$

It is known that, for $p \geq 1$,

$$(1+\|w\|_p)^{1/p} \leq \|S\|_{L^p_{\mathrm{dec}}(w)} \leq 1+\|w\|_p.$$

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Minimal r.i. spaces

Let X be a rearrangement invariant space (r.i.) in $\mathbb{R}^n$; that is, a Banch function space satisfying:

1. $\chi_E \in X$, for every $0 < |E| < \infty$.
2. If $0 < |E| < \infty$ and $f \in X$, then $\left| \int_E f(x)\,dx \right| \leq C_E \|f\|_X$.
3. If $0 \leq f_n \uparrow f$, then $\|f_n\|_X \uparrow \|f\|_X$.
4. If f and g are equimeasurable, then $\|f\|_X = \|g\|_X$.

Equimeasurable means with the same distribution function:

$$\lambda_f(t) = |\{|f| > t\}| = |\{|g| > t\}| = \lambda_g(t).$$

Examples:

- Lebesgue spaces L^p.
- Weighted Lorentz spaces $\Lambda^p(w)$.
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The fundamental function of X is defined as

$$\varphi_X(t) = \|\chi_E\|_X, \qquad |E| = t.$$

This function plays an important role in the theory. For example if we define the (minimal) Lorentz space:

$$\Lambda(X) = \Lambda_{\varphi_X} = \left\{ f; \|f\|_{\Lambda(X)} = \int_0^\infty f^*(t)\, d\varphi_X(t) < \infty \right\},$$

and the (maximal) Marcinkiewicz space:

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then

$$\Lambda(X) \subset X \subset M(X).$$

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We are now interested in studying the norm of $S - \mathrm{Id}$ on an r.i. space X, for the n-dimensional Hardy operator, acting on radially decreasing functions f:

$$S_n f(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f(y)\, dy.$$

Using that

$$S_n f(x) - f(x) = \frac{1}{v_n |x|^n} \int_{f(x)}^{\infty} \lambda_f(t)\, dt,$$

(where $\lambda_f(t) = |\{x : |f(x)| > t\}|$), then

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We are now interested in studying the norm of $S - \mathrm{Id}$ on an r.i. space X, for the n-dimensional Hardy operator, acting on radially decreasing functions f:

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Hence, it is natural to study the class of functions for which the right-hand side is finite [Studia Math., 2010], that we call $R(X)$:

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It can be proved that the norm on $R(X)$ is equal to:

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Theorem (Rodríguez – S. [PEdMS, 2012])

- TFAE:
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- It is well-known that $\Lambda^2 = \Lambda \circ \Lambda = \Lambda$ and hence $\Lambda \circ R = R$.
  However there are examples showing that $R^2 = R \circ R \neq R$:

  If $\Phi(t) = t/\log(1+t)$, then

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  In fact, all the examples we know satisfy one of the following 3 possibilities:

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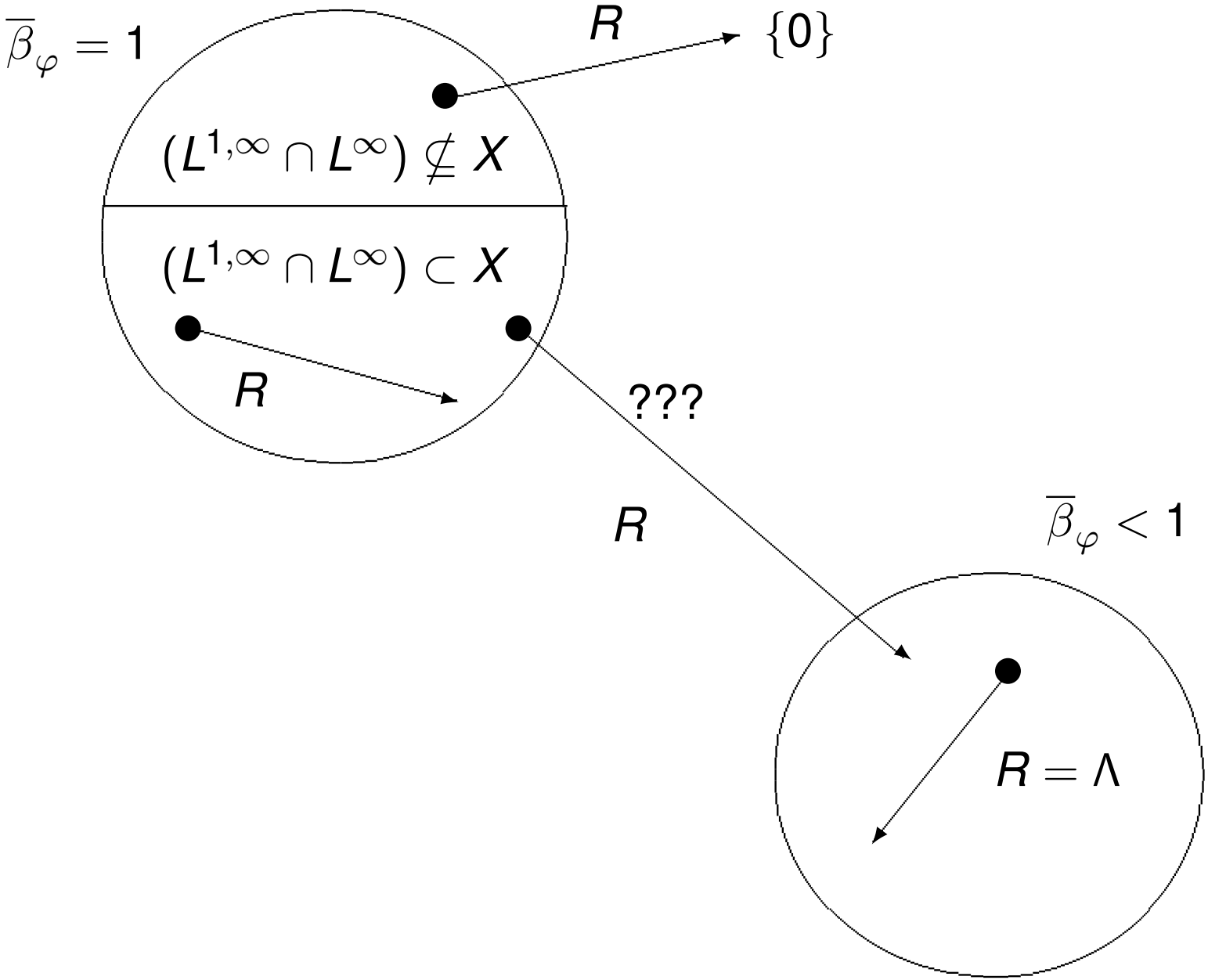
Remarks

- We know that if $\overline{\beta}_X < 1$, then $R(X) = \Lambda(X)$ and hence $\overline{\beta}_X = \overline{\beta}_{R(X)} < 1$, but we do not know whether the converse is true:

  If $\overline{\beta}_{R(X)} < 1$,

  $$\overline{\beta}_X = \overline{\beta}_{R(X)}?$$

  Or, does there exist X with $\overline{\beta}_X = 1$ but $\overline{\beta}_{R(X)} < 1$?


$\overline{\beta}_\varphi = 1$
R
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- By definition, if f is decreasing,

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and the inequality is sharp.

In particular, if $1 < p < \infty$, $1 \leq q \leq p$, then

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Range of R

Since R is a monotone functor, then for every X r.i.,

$$R(X) = \Lambda_{W_X} \subset \Lambda_\Psi = R(L^1 + L^\infty),$$

and hence, $W_X(t) \geq C\Psi(t) = t\log(1 + 1/t)$.

We are now interested in studying when a given Lorentz space Λ_φ belongs to the range of R; i.e., there exists an X such that

$$R(X) = \Lambda_\varphi.$$

A necessary condition for this to happen is that

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Σας ευχαριστώ για την προσοχή σας!

Thanks for your attention!

¡Gracias por vuestra atención!