Composition operators on B^1

Michael Papadimitrakis, joint with Themis Mitsis

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The Besov-1 space

Δ is the unit disc in $\mathbb{C}$ and the Möbius functions

$$\mu\phi_a(z) = \mu\tfrac{a-z}{1-\overline{a}z}\,, \qquad a \in \Delta, |\mu| = 1$$

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The Besov-1 space, B^1 , consists of all functions f analytic on Δ which can be written as

$$f = \textstyle\sum_{n=1}^{+\infty} \lambda_n \phi_{a_n}\,, \qquad a_n \in \overline{\Delta}, \textstyle\sum_{n=1}^{+\infty} |\lambda_n| < +\infty.$$

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is a norm on B^1 under which we have $\|\phi_a\|_{B^1} = 1$ for all $a \in \overline{\Delta}$.

It is easily proved that, under this norm, B^1 is a Banach space *invariant* under right-composition by Möbius functions and that it is the *minimal* space with this property.

The Besov-1 space

Equivalently, B^1 consists of all f analytic on Δ for which $\iint_\Delta |f''(z)|\, dA(z) < +\infty$, where dA is the normalized Lebesgue measure on Δ. We have the following equivalence between norms:

$$\|f\|_{B^1} \asymp |f(0)| + |f'(0)| + \iint_\Delta |f''(z)|\, dA(z).$$

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One direction of this equivalence comes from the formulae

$$\phi_a{}'(z) = \tfrac{|a|^2-1}{(1-\overline{a}z)^2}\,, \qquad \phi_a{}''(z) = 2\overline{a}\tfrac{|a|^2-1}{(1-\overline{a}z)^3}$$

and the resulting

$$\iint_\Delta |\phi_a{}''(z)|\, dA(z) \lesssim 1.$$

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The other direction comes from an appropriate discretization of the formula

$$f(z) = f(0) + f'(0)z - \iint_\Delta \frac{1}{\overline{a}}\, f''(a)\phi_a(z)\, dA(a).$$

Other Möbius invariant spaces

B^p $(1 < p < +\infty)$, the Besov-p space, consists of all f analytic in Δ with the norm

$$\begin{aligned}\|f\|_{B^p} &= |f(0)| + \iint_\Delta |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z) \\ &\asymp_p |f(0)| + |f'(0)| + \iint_\Delta |f''(z)|^p (1-|z|^2)^{p-1}\, dA(z).\end{aligned}$$

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B is the Bloch space, consisting of all b analytic on Δ with

$$\sup_{z\in\Delta}(1-|z|^2)|b'(z)| < +\infty.$$

B_0 is the little-Bloch space, consisting of all b analytic on Δ with

$$\lim_{z\in\Delta, |z|\to 1}(1-|z|^2)|b'(z)| = 0.$$

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It is true that

$$B^1 \subseteq B^{p_1} \subseteq B^{p_2} \subseteq B_0 \subseteq B$$

for $1 < p_1 < p_2 < +\infty$ and B^1 is the “limit” of B^p as $p \to 1$.

Other Möbius invariant spaces are H^∞, the disc algebra A, the $BMOA$ and $VMOA$, all containing B^1 .

Duality

For all $f \in B^1$ and all $b \in B$ we define the sesquilinear product

$$\langle f, b \rangle = f(0)\overline{b(0)} + f'(0)\overline{b'(0)} + \iint_\Delta \frac{1}{\bar{z}} f''(z)(1-|z|^2)\overline{b'(z)}\, dA(z).$$

Through this product one may easily prove the dualities

$$B^{1*} \cong B, \qquad {B_0}^* \cong B^1 .$$

Composition operators: boundedness

For analytic $\psi : \Delta \to \Delta$ we define the *composition operator*

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We have the following sequence of equivalent condtitions:

$$\sup_{a\in\Delta}(1-|a|^2)\iint_\Delta \left|2\overline{a}\frac{\psi'(z)^2}{(1-\overline{a}\psi(z))^3} + \frac{\psi''(z)}{(1-\overline{a}\psi(z))^2}\right| dA(z) < +\infty,$$

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$$\sup_I \left(\frac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) + \frac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z)\right) < +\infty,$$

where I is the typical arc of the unit circle and $S = S(I)$ is the Carleson square with I as its base.

Composition operators: compactness

It is also known that $C_\psi : B^1 \to B^1$ is *compact* if and only if

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We also have the corresponding sequence of equivalent condtitions, with the $\sup_{a\in\Delta}$ replaced by $\lim_{a\in\Delta,|a|\to 1}$ and with $\sup_I$ replaced by $\lim_{|I|\to 0}$.

It is somewhat strange that all these conditions are equivalent to

$$\sup_{z\in\Delta} |\psi(z)| < 1.$$

Composition operators: closed range

As a general fact, $C_\psi : B^1 \to B^1$ has *closed range* if and only if

$$\|C_\psi f\|_{B^1} \geq c\|f\|_{B^1}\,, \qquad f \in B^1\,,$$

for some constant $c > 0$ independent of $f \in B^1$.

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We shall prove that the condition

$$\liminf_{|I|\to 0} \sup_{a\in T(S)}(1-|a|^2) \iint_{\psi^{-1}(S)} \left|2\overline{a}\frac{\psi'(z)^2}{(1-\overline{a}\psi(z))^3} + \frac{\psi''(z)}{(1-\overline{a}\psi(z))^2}\right| dA(z) > 0,$$

where $T(S)$ is the inner half of the Carleson square S, is *necessary* for C_ψ to have closed range.

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where $T(S)$ is the inner half of the Carleson square S, is *necessary* for C_ψ to have closed range.

The last condition is equivalent to

$$\liminf_{|I|\to 0} \left(\frac{1}{|S|}\iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) + \frac{1}{|I|}\iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z)\right) > 0.$$

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If $a \in T(S)$ and $\psi(z) \in S$, then $1 - |a|^2 \asymp |I|$ and $|1 - \overline{a}\psi(z)| \asymp |I|$. Therefore, the first condition trivially implies the second.

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Now, suppose that for some sequence of I's with $|I| \to 0$ we have

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This implies

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and, if moreover $|a_1 - a_2| \asymp |I|$, then we get

$$\tfrac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2 \, dA(z) \to 0, \qquad \tfrac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)| \, dA(z) \to 0.$$

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Proof of the necessity of the second condition.

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Suppose that for some sequence of I's with $|I| \to 0$ we have

$$\delta_I := \frac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2 \, dA(z) + \frac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)| \, dA(z) \to 0.$$

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For each I (in our sequence) we consider a *much smaller* (?) arc I' so that I and I' have the same center. We consider the corresponding $S = S(I)$ and $S' = S(I')$ and, also, two $a_1, a_2 \in T(S')$ such that $|a_1 - a_2| \asymp |I'|$ and $1 - |a_1|^2 = 1 - |a_2|^2 \asymp |I'|$.

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Now we take

$$f = \phi_{a_1} - \phi_{a_2} \, .$$

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Now we take

$$f = \phi_{a_1} - \phi_{a_2} \, .$$

It is easy to see that

$$\|f\|_{B^1} \asymp 1$$

and we shall prove that

$$\|C_\psi f\|_{B^1} \to 0.$$

Composition operators: closed range

$$f'(z) = \frac{|a_1|^2-1}{(1-\overline{a_1}z)^2} - \frac{|a_2|^2-1}{(1-\overline{a_2}z)^2} = (|a_1|^2-1)(\overline{a_1}-\overline{a_2})\frac{2-(\overline{a_1}+\overline{a_2})z}{(1-\overline{a_1}z)^2(1-\overline{a_2}z)^2}$$

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where a is any of a_1 or a_2.

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$$|f''(z)| \asymp \frac{|I'|^2}{|1-\overline{a}z|^4}\,.$$

We have

$$\|C_\psi f\|_{B^1} \asymp |f(\psi(0))| + |f'(\psi(0))||\psi'(0)| + A + B,$$

where

$$A := \iint_{\psi^{-1}(S)} |f''(\psi(z))\psi'(z)^2 + f'(\psi(z))\psi''(z)|\, dA(z),$$

$$B := \iint_{\Delta\setminus\psi^{-1}(S)} |f''(\psi(z))\psi'(z)^2 + f'(\psi(z))\psi''(z)|\, dA(z).$$

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Trivially:

$$|f(\psi(0))| + |f'(\psi(0))||\psi'(0)| \asymp |I'| + |I'|^2 \to 0.$$

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A &\lesssim |I'|^2 \iint_{\psi^{-1}(S)} \tfrac{|\psi'(z)|^2}{|1-\overline{a}\psi(z)|^4}\, dA(z) + |I'|^2 \iint_{\psi^{-1}(S)} \tfrac{|\psi''(z)|}{|1-\overline{a}\psi(z)|^3}\, dA(z) \\
&\lesssim \tfrac{|I|^2}{|I'|^2} \tfrac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) + \tfrac{|I|}{|I'|} \tfrac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z) \\
&\lesssim \frac{|I|^2}{|I'|^2}\, \delta_I
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and

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B &\lesssim |I'|^2 \iint_{\Delta\setminus\psi^{-1}(S)} \tfrac{|\psi'(z)|^2}{|1-\overline{a}\psi(z)|^4}\, dA(z) + |I'|^2 \iint_{\Delta\setminus\psi^{-1}(S)} \tfrac{|\psi''(z)|}{|1-\overline{a}\psi(z)|^3}\, dA(z) \\
&\lesssim \tfrac{|I'|}{|I|}\,(1-|a|^2) \left(\iint_{\Delta} \tfrac{|\psi'(z)|^2}{|1-\overline{a}\psi(z)|^3}\, dA(z) + \iint_{\Delta} \tfrac{|\psi''(z)|}{|1-\overline{a}\psi(z)|^2}\, dA(z) \right) \\
&\lesssim \frac{|I'|}{|I|}\,.
\end{aligned}$$

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Therefore,

$$A + B \lesssim \frac{|I|^2}{|I'|^2}\,\delta_I + \frac{|I'|}{|I|}\,.$$

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Now let

$$|I'| = \sqrt[3]{\delta_I}\,|I|$$

and conclude that

$$A + B \lesssim \sqrt[3]{\delta_I} \to 0.$$

“Closed range” results for other spaces

1. Zorboska “Composition operators with closed range” (1994)

Equivalent conditions. Weighted Bergman spaces A^2_α defined by $\iint_\Delta |f(z)|^2(1-|z|^2)^\alpha \, dA(z) < +\infty$ and for the H^2 space $(\iint_\Delta |f'(z)|^2(1-|z|^2)^\alpha \, dA(z) < +\infty)$.

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2. Jovovic, McCluer “Composition operators on Dirichlet spaces” (1997)

Weighted Dirichlet spaces $\mathcal{D}_\alpha$ defined by $\iint_\Delta |f'(z)|^2(1-|z|^2)^\alpha\,dA(z) < +\infty$. ($\mathcal{D}_0 = \mathcal{D}, \mathcal{D}_1 = H^2, \mathcal{D}_2 = A^2$).

For $\mathcal{D}$. Sufficient (not necessary) condition: $|\psi(\Delta)\cap S| \geq c|S|$. Necessary condition: $\frac{1}{|S|}\iint_S n_\psi(w)\,dA(w) \geq c$. Proved not sufficient by Luecking.

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4. Akeroyd, Ghatage, Tjani “Closed range composition operators on A^2 and the Bloch space” (2010)

Equivalent condition(s). For the Bloch space.