Spectra of integration operators on Hardy spaces

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Joint work with A. Aleman

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- For $0 < p < \infty$ $\alpha > -1$, the *the standard weighted Bergman space* A^p_α consists of those $f \in H(\mathbb{D})$ for which

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- $f \in \mathcal{B}_0$ if $\lim_{|z| \to 1^-} (1 - |z|^2)|f'(z)| = 0$.

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- Previous result Young (2004) Aleman-Persson (2010):
  If g' is a rational function with $g \in BMOA$, then $\sigma(T_g|H^p)$ is a union of closed disks,
  $$\sigma(T_g|H^p) = \{0\} \cup \overline{\{\lambda \neq 0 : e^{g/\lambda} \notin H^p\}}, \quad \text{if } p \geq 1.$$

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$$f(z) = R_{\lambda,g} h(z) = \frac{h(0)}{\lambda} e^{\frac{g(z)}{\lambda}} + \frac{e^{\frac{g(z)}{\lambda}}}{\lambda} \int_0^z e^{-\frac{g(\xi)}{\lambda}} h'(\xi)\, d\xi, \quad z \in \mathbb{D}.$$

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- If $\tilde{R}_{\lambda,g} h(z) = e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\xi)}{\lambda}} h'(\xi)\, d\xi$,

$$R_{\lambda,g} \text{ is a bounded} \Leftrightarrow e^{\frac{g(z)}{\lambda}} \in H^p \text{and } \tilde{R}_{\lambda,g} \text{ is a bounded.}$$

- $\tilde{R}_{\lambda,g}$ is bounded on H^p if

$$||\tilde{R}_{\lambda,g}(h)||_{H^p}^p = \int_{\mathbb{T}} \exp\left(p\, Re\left(\frac{g(\zeta)}{\lambda}\right)\right) \left|\int_0^\zeta e^{-\frac{g(\xi)}{\lambda}} h'(\xi)\, d\xi\right|^p dm(\zeta)$$

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- If $H(z) = \int_0^z e^{-\frac{g(\xi)}{\lambda}} h'(\xi)\,d\xi$, so that $h' = e^{\frac{g}{\lambda}} H'$, $\omega(e^{i\theta}) = \exp\left(p\,Re\left(\frac{g(e^{i\theta})}{\lambda}\right)\right)$ and $W(z)$ outer function with $|W(e^{i\theta})| = \omega(e^{i\theta})$

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$$\int_{\mathbb{T}} |H(\zeta)|^p \omega(\zeta)\, dm(\zeta) \leq C \int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} |W|^{2/p} |H'|^2 dA\right)^{p/2} dm(\zeta)\,.$$

Reformulation *(Aleman-Peláez (2011))*

Assume that $\lambda \in \mathbb{C} \setminus \{0\}$, $0 < p < \infty$ *and* $g \in BMOA$. *Then, the following assertions are equivalent:*

(i) $\lambda \in \rho(T_g | H^p)$.

(ii) $e^{\frac{g}{\lambda}} \in H^p$ *and for any* $f \in H(\mathbb{D})$ *with* $f(0) = 0$

$$\int_{\mathbb{T}} |f(\zeta)|^p \omega(\zeta)\, dm(\zeta) \sim \int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} |W|^{2/p} |f'|^2 dA \right)^{p/2} dm(\zeta)$$

where $\omega(e^{i\theta}) = \exp\left(p\, Re\left(\frac{g(e^{i\theta})}{\lambda} \right)\right)$ *and* $W(z)$ *is the outer function with* $|W(e^{i\theta})| = \omega(e^{i\theta})$.

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Proposition *(Aleman-Peláez (2011))*

The following are equivalent:
(i) ω *satisfies* $\mathcal{A}_\infty$.
(ii) If $W(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log \omega(\zeta) dm(\zeta)\right)$, *then* $\log W \in \mathcal{B}$ *and*

$$|W(z)| \sim \frac{1}{m(I_z)} \int_{I_z} \omega dm\,, \quad z \in \mathbb{D}.$$

(iii) There exists $\eta > 1$ *such that*

$$(1-|z|^2)^{\eta-1} \int_{\mathbb{T}} \frac{\omega(\zeta)}{|\zeta - z|^\eta} dm(\zeta) \lesssim |W(z)|\,, \quad z \in \mathbb{D}\,.$$

- Problems: Which weights do satisfy the estimates

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- Sufficient condition?

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Theorem *(Aleman-Peláez (2011))*

If ω satisfies the conformal invariant $\mathcal{A}_\infty$ condition and $p>0$ then for analytic functions f in $\mathbb{D}$

$$\|f\|_{p,\omega}^p\sim|f(0)|^p+\int_{\mathbb{T}}S^p_{\omega,p,f}\,dm\sim|f(0)|^p+I_{\omega,p}(f).$$

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Lemma *(Treil-Volberg-Zheng (1997))*

If the weight ω satisfies the conformal invariant $\mathcal{A}_\infty$ condition then for analytic functions g in $\mathbb{D}$ and $p>0$, we have

$$\int_{\mathbb{D}}|g(z)|^p\frac{|\nabla w(z)|^2}{w^2(z)}\log\frac{1}{|z|}\,dA(z)\lesssim\|g\|^p_{H^p}\,.$$

Main Theorem *(Aleman-Peláez (2011))*

For a weight ω *on* $\mathbb{T}$ *and* $0 < p < \infty$ *the following are equivalent:*
(i) ω *satisfies* $\mathcal{A}_\infty$*-condition,*
(ii) For analytic functions f *in* $\mathbb{D}$ *we have*

$$\|f\|_{p,\omega}^p \sim |f(0)|^p + \int_{\mathbb{T}} S_{\omega,p,f}^p \, dm \,,$$

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Lemma *(Gundy-Wheeden (1974))*

Assume that $0 < p < \infty$ *and* ω *satisfies* $\mathcal{A}_\infty$. *Then for any harmonic function* u *in* $\mathbb{D}$ *with* $u(0) = 0$

$$\int_{\mathbb{T}} (u^*)^p(\zeta)\omega(\zeta)\,dm(\zeta)$$
$$\lesssim \int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(z)} |\nabla u|^2 dA \right)^{p/2} \omega(\zeta)\,dm(\zeta)$$

where u^* *is the nontangential maximal function.*

Proof of $(i) \Rightarrow (iii)$ for $0 < p < 2$.

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By G-W's result

$$\int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(z)} |\nabla\, Im\, f|^2 dA \right)^{p/2} \omega(\zeta)\, dm(\zeta)$$
$$\gtrsim \int_{\mathbb{T}} (Im\, f)^p(\zeta)\omega(\zeta)\, dm(\zeta)$$

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$$\int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(z)} |\nabla\, Re\, f|^2 dA \right)^{p/2} \omega(\zeta)\, dm(\zeta)$$
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Since $|\nabla\, Im\, f| \sim |\nabla\, Re\, f| \sim |f'|$, it suffices to prove

$$I_{\omega,p}(f) \gtrsim \int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} |f'|^2\, dA \right)^{p/2} \omega(\zeta)\, dm(\zeta). \tag{1}$$

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\begin{aligned}
&\int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} |f'|^2 \, dA \right)^{p/2} \omega(\zeta) \, dm(\zeta) \\
&\le \left(\int_{\mathbb{T}} (f^*)^{p-2} \left(\int_{\Gamma_\sigma(\zeta)} |f'|^2 \, dA \right) \omega(\zeta) \, dm(\zeta) \right)^{p/2} \left(\int_{\mathbb{T}} (f^*)^p \omega(\zeta) \, dm(\zeta) \right)^{1-p/2} \\
&\lesssim \left(\int_{\mathbb{T}} (f^*)^{p-2} \left(\int_{\Gamma(\xi)} |f'|^2 \, dA \right) \omega(\zeta) \, dm(\zeta) \right)^{p/2} \\
&\times \left(\int_{\mathbb{T}} (((Re\, f)^*)^p(\zeta) + ((Im\, f)^*)^p(\zeta) \, dm(\zeta) \right)^{1-p/2} \\
&\lesssim \left(\int_{\mathbb{T}} (f^*)^{p-2} \left(\int_{\Gamma_\sigma(\zeta)} |f'|^2 \, dA \right) \omega(\zeta) \, dm(\zeta) \right)^{p/2} \\
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Next,

$$\int_{\mathbb{T}} (f^*)^{p-2} \left(\int_{\Gamma_\sigma(\zeta)} |f'|^2 \right) \omega(\zeta)\, dm(\zeta)\, dA \gtrsim \int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} |f'|^2\, dA \right)^{p/2} \omega(\zeta)\, dm(\zeta). \tag{2}$$

Next,

$$\begin{aligned} I_{\omega,p}(f) &= \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 |W(z)| \log \frac{1}{|z|}\, dA(z) \\ &\asymp \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left(\int_{I_z} \omega(\zeta)\, dm(\zeta) \right) dA(z) \\ &= \int_{\mathbb{T}} \int_{\Gamma_\sigma(\zeta)} |f(z)|^{p-2} |f'(z)|^2\, \omega(\zeta)\, dm(\zeta) \\ &\geq \int_{\mathbb{T}} (f^*)^{p-2} \left(\int_{\Gamma_\sigma(\zeta)} |f'|^2 \right) \omega(\zeta)\, dm(\zeta), \end{aligned}$$

which together with (2) gives (1).

- Note : If for harmonic functions u in $\mathbb{D}$ and $\zeta \in \mathbb{T}$

$$S^h_{\omega,p,u}(z) = \left(\int_{\Gamma_\sigma(z)} |\nabla u|^2 |W|^{2/p} dA \right)^{1/2}$$

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Corollary *(Aleman-Peláez (2011))*

The weight ω satisfies $\mathcal{A}_p$, $p > 1$, if and only if

$$|u(0)|^p + \int_{\mathbb{T}} (S^h_{\omega,p,u})^p dm \sim \|u\|^p_{p,\omega}$$

holds for all Poisson integrals u of integrable functions on $\mathbb{T}$.

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Theorem *(Aleman-Peláez (2011))*

Assume that $\lambda \in \mathbb{C} \setminus \{0\}$, $p \in (0, \infty)$, *and* $g \in BMOA$. *Then, the following assertions are equivalent:*

(i) $\lambda \in \rho(T_g | H^p)$.

(ii) $e^{\frac{g}{\lambda}} \in H^p$ *and* $\omega(e^{i\theta}) = \exp\left(p \, Re\left(\frac{g(e^{i\theta})}{\lambda}\right)\right)$ *satisfies the* $\mathcal{A}_\infty$ *condition.*

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(i) If ω satisfies $\mathcal{A}_\infty$ then there exists $\varepsilon > 0$ such that $|W^\alpha|$ is $\mathcal{A}_\infty$ for all $\alpha \in \mathbb{C}$ with $|\alpha - 1| < \varepsilon$, and there exist $p > 1$ such that ω satisfies $\mathcal{A}_p$.
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$\omega = \exp(Re \log W)$ and apply the previous theorem to conclude that 1 belongs to the resolvent set of $T_{\log W}$.
It is an open set, $\lambda I - T_{\log W}$ is invertible when λ is sufficiently close to 1.
$\exp\left(Re\left(\frac{\log W}{\lambda}\right)\right)$ satisfies $\mathcal{A}_\infty$ when λ is sufficiently close to 1.

Corollary *(Aleman-Peláez (2011))*

(i) If ω satisfies $\mathcal{A}_\infty$ then there exists $\varepsilon > 0$ such that $|W^\alpha|$ is $\mathcal{A}_\infty$ for all $\alpha \in \mathbb{C}$ with $|\alpha - 1| < \varepsilon$, and there exist $p > 1$ such that ω satisfies $\mathcal{A}_p$.
(ii) In particular, if ω satisfies $\mathcal{A}_p$ then there exists $\varepsilon' > 0$ such that $|W^\alpha|$ is $\mathcal{A}_p$ for all $\alpha \in \mathbb{C}$ with $|\alpha - 1| < \varepsilon'$.

Proof.

It suffices to prove the first part.
$\omega = \exp(Re \log W)$ and apply the previous theorem to conclude that 1 belongs to the resolvent set of $T_{\log W}$.
It is an open set, $\lambda I - T_{\log W}$ is invertible when λ is sufficiently close to 1.
$\exp\left(Re\left(\frac{\log W}{\lambda}\right)\right)$ satisfies $\mathcal{A}_\infty$ when λ is sufficiently close to 1.
The spectrum of $T_{\log W}$ is bounded, the resolvent set must intersect the negative real-axis, which gives the second assertion. $\square$