Derivatives of Blaschke products and Bergman spaces with normal weights

(Joint work with A. Aleman)

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Blaschke products

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$: unit disk.

$\{a_n\}$: a sequence of points in $\mathbb{D}$.

Blaschke condition: $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$.

The corresponding Blaschke product B is defined as

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}.$$

B is analytic in $\mathbb{D}$, bounded by one, and has radial limits of modulus one almost everywhere on the unit circle.

Blaschke products are fundamental in the theory of Hardy spaces (being isometric zero-divisors).

Question. Study the behavior of B', *e.g.*, its membership in different function spaces.

Some contributors

Rudin, Piranian: initial examples (50's, 60's).

Ahern, Clark, Cohn, Protas, Kim (70's, 80's): systematic study of the membership of B' in Hardy and Bergman spaces of the disk. Conditions on zeros.

Kutbi, Fricain, Mashreghi (2000-2010): recent contributions to the study of integral means of the derivative (on circles centered at the origin).

Girela, Peláez, V. (2005-10): further systematic study: zeros in a Stolz angle and nontangential approach regions, best exponents of integrability for interpolating Blaschke products.

Aleman, V. (2010): membership of B' in Bergman spaces with normal weights, relation with interpolation and duality results.

Bergman spaces

$z = re^{i\theta} \in \mathbb{D}$, $dA(z) = \frac{1}{\pi} r\,dr\,d\theta$.

For $0 < p < \infty$, we say that $f \in L^p_a$ (the Bergman space) if

$$\int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\,.$$

$p > q$ implies $L^p_a \subset L^q_a$.

Schwarz-Pick Lemma:

$$|B'(z)| \leq \frac{1 - |B(z)|^2}{1 - |z|^2}$$

easily implies that $B' \in \cap_{0<p<1} L^p_a$.

Rudin (1955): there exists B whose derivative is not in L^1_a.

Piranian (1968): an explicit example.

Normal weights

Weight: a positive measurable function $w : \mathbb{D} \to [0, +\infty)$, usually integrable.

A weight is called *radial* if $w(z) = w(|z|)$ for all z in $\mathbb{D}$.

A radial weight is *normal* (Shields, Williams) if there exist a, $b \in \mathbb{R}$ and $r_0 \in (0,1)$ such that

$$\frac{w(r)}{(1-r)^a} \nearrow \infty \quad \text{when} \quad r > r_a \,,$$

$$\frac{w(r)}{(1-r)^b} \searrow 0 \quad \text{when} \quad r > r_b \,.$$

Let $a_w = \inf a$ and $b_w = \sup b$. Then $a_w \geq b_w$.

Example. Let $-1 < \alpha$, $\beta < \infty$, and consider

$$w(r) = (1-r)^\alpha \log^\beta \frac{1}{1-r} \,.$$

Then $a_w = b_w = \alpha$.

Weighted Bergman spaces

We consider only integrable normal weight w. For example, this is guaranteed if we assume that $b_w > -1$.

Define the *weighted Bergman space* $L^p_a(w)$, $p > 0$, to consists of all analytic functions f in $\mathbb{D}$ such that

$$\|f\|^p_{p,w} = \int_{\mathbb{D}} |f|^p w dA < \infty \,.$$

From now on, assume: the weight w is both normal and integrable in $[0, 1)$.

Such spaces generalize the standard Bergman spaces with

$$w(z) = (\alpha + 1)(1 - r^2)^{\alpha}\,, \quad -1 < \alpha < \infty \,.$$

For any normal and integrable weight it can be shown that $f \in L^p_a(w)$ implies

$$|f(z)| \leq C(w(|z|)^{-1/p}(1 - |z|^2)^{-2/p}\|f\|_{p,w}\,,$$

for some constant $C > 0$ and all $z \in \mathbb{D}$.

When $b_w > -1$, this implies

$$\sup_{\|f\|_{p,w} \leq 1} |f(z)| \asymp (w(|z|))^{-1/p}(1 - |z|^2)^{-2/p},$$

as $|z| \to 1^-$.

Interpolating sequences

A sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{D}$ is called an *interpolating sequence* for $L^p_a(w)$ if for every $(a_n) \in \ell^p$ there exists $f \in L^p_a(w)$ such that

$$f(z_n)w(|z_n|)^{1/p}(1 - |z_n|^2)^{2/p} = a_n\,,$$

for all n.

If $p = 2$, it is well known that interpolating sequences for Hardy spaces are interpolating for $L^2_{a,w}$ as well. This follows from a well-known result that the only pointwise multipliers from $L^2_{a,w}$ into itself are the H^∞ functions and from certain interpolation results.

This can be extended for the other values of $p > 1$ but requires different techniques.

Theorem. If $b_w > -1$, then every interpolating sequence for H^∞ is interpolating for $L^p_a(w),\ p > 1$.

Proof. If $\frac{1}{p} + \frac{1}{q} = 1$, let $\gamma > \frac{a_w}{p} - \frac{1}{q}$.

Can show that

$$\left(\int_r^1 w(t)dt\right)\left(\int_r^1 (1-t)^{\gamma q} w^{-q/p}(t)dt\right)^{p/q}$$

$$\leq C(1-r)^{p\gamma+p}.$$

This implies that $w(|z|)(1-|z|)^{-\gamma}$ belongs to the Békollé class $B_p(\gamma)$.
Equivalently, $v(z) = w^{-q/p}(z)(1-|z|)^{(q-1)\gamma}$ belongs to $B_q(\gamma)$.

By a result of Békollé (1982) it follows that the sublinear operator P_γ defined by

$$P_\gamma f(z) = \int_{\mathbb{D}} \frac{(1-|\zeta|)^\gamma}{|1-\bar{\zeta}z|^{\gamma+2}} |f(\zeta)| dA(\zeta)$$

is bounded from $L^q((1-|z|)^{q\gamma} w^{-q/p} dA)$ into itself.

Luecking's duality theorem (1985) identifies the dual of $L^p_a(w)$ with $L^q_a(v)$ via a standard weighted Bergman pairing. More precisely, it says that the norm of a function $f \in L^p_a(w)$ is equivalent to

$$\sup\left\{\left|\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\gamma dA\right| : \right.$$
$$\left. g \in L^q_a(w)\,, \|g\|_{q,v} \le 1\right\}\,.$$

Let $(z_n)_{n=1}^\infty$ be an interpolating sequence for H^∞, let B be the Blaschke product with zeros z_1, z_2, ... and let $a = (a_n) \in \ell^p$.

We now exhibit an explicit solution f_a of the interpolation problem defined above:

$$f_a(z) = \sum_n a_n(w(|z_n|))^{-1/p}(1-|z_n|^2)^{-2/p} \times$$
$$\times \frac{(1-|z_n|^2)^{\gamma+1}B(z)}{(1-\overline{z}_n z)^{\gamma+1}(z-z_n)B'(z_n)}\,.$$

In order to see this, it suffices to show that

$$\|f_a\|_{p,w} \le C\|a\|_{\ell^p}\,,$$

for some constant C and all sequences a in ℓ^p with finitely many nonzero terms.

Denote by B_n the Blaschke product with zeros $(z_k)_{k\neq n}$, taking multiplicities into account. Recall that the values $|B_n(z_n)|$ are bounded away from zero by the separation assumption. By the duality relation,

$$\begin{aligned}
& \|f_a\|_{p,w} \leq \\
& \sup_{\|g\|_{q,v}\leq 1} \sum_n \frac{|a_n|}{|B_n(z_n)|} w^{-\frac{1}{p}}(|z_n|)(1-|z_n|)^{\gamma+\frac{2}{q}} \times \\
\times \quad & \int_{\mathbb{D}} \frac{|g(z)||B_n(z)|(1-|z|)^{\gamma}}{|1-\overline{z}_n z|^{\gamma+2}} dA(z) \\
\leq \quad & C \sup_{\|g\|_{q,v}\leq 1} \sum_n |a_n| w^{-\frac{1}{p}}(|z_n|)(1-|z_n|)^{\gamma+\frac{2}{q}} \times \\
\times \quad & P_\gamma g(z_n) \\
\leq \quad & C\|a\|_{\ell^p} \sup_{\|g\|_{q,v}\leq 1} \\
& \left(\sum_n w^{-\frac{q}{p}}(|z_n|)(1-|z_n|)^{q\gamma+2} P_\gamma^q g(z_n)\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $(z_n)_{n=1}^{\infty}$ is uniformly separated, there is a fixed $R \in (0,1)$ such that the disks

$$\Delta_n = \{\zeta \in \mathbb{D} \colon |\zeta - z_n| < R\,(1 - |z_n|)\}\,,$$

are pairwise disjoint.

One can also see that $P_\gamma g$ is "almost constant" on these disks, *i.e.*, there exists $c_0 > 0$ independent of g such that

$$c_0^{-1} P_\gamma g(z) \leq P_\gamma g(\zeta) \leq c_0 P_\gamma g(z)\,, \qquad z, \zeta \in \Delta_n\,.$$

Normal weights also behave nicely on such disks, so all together this yields

$$\begin{aligned}
\sup_{\|g\|_{q,v}\leq 1} \quad & \sum_n w^{-q/p}(|z_n|)(1-|z_n|)^{q\gamma+2} \times \\
& \times P_\gamma^q g(z_n) \\
\leq \qquad & C \sup_{\|g\|_{q,v}\leq 1} \sum_n w^{-q/p}(|z_n|) \times \\
& \times (1-|z_n|)^{q\gamma} \int_{\Delta_n} P_\gamma^q g(z)\,dA(z) \\
\leq \qquad & C \sup_{\|g\|_{q,v}\leq 1} \sum_n \int_{\Delta_n} P_\gamma^q g(z) w^{-\frac{q}{p}}(z) \times \\
& \times (1-|z|)^{q\gamma} dA(z)
\end{aligned}$$

$$\leq \sup_{\|g\|_{q,v}\leq 1} \int_{\cup_n \Delta_n} P_\gamma^q g(z) w^{-\frac{q}{p}}(z)(1-|z|)^{q\gamma} dA(z)\,.$$

and the result follows.

Integrability of B'

For a nonconstant analytic function f in $\mathbb{D}$, a point ζ in $\mathbb{D}$, and $u : \mathbb{D} \to [0, +\infty)$, use the abbreviated notation

$$\sum_{f(z)=\zeta} u(z)$$

to denote the summation over the set $f^{-1}(\{\zeta\})$, taking multiplicities into account.

Well-known:

$$\|B'\|_1 \leq c \sum_{B(z)=0} (1-|z|) \log \frac{1}{1-|z|}\,,$$

We complement this (and other results).

Theorem (H.O. Kim). Suppose that a Blaschke product B satisfies

$$\sum_{B(z)=0}^{\infty} (1-|z|)^{2-p} < \infty\,, \quad 1 < p < 2\,.$$

Then $B' \in A^p$.

Theorem (Girela-Peláez-V.) If B is an interpolating Blaschke product then

$$\int_{\mathbb{D}} |B'(z)|^p dA(z) \geq C \sum_{B(z)=0} (1-|z|)^{2-p}\,.$$

In particular, if the series on the right diverges, then $B' \notin A^p$.

A sequence $\{a_n\}_n$ is said to be *separated* (with *constant of separation* δ) if

$$\inf_{k\neq n} \left| \frac{a_k - a_n}{1 - \overline{a}_k a_n} \right| = \delta > 0\,.$$

Theorem. Let $a_w < 2p-2,\ b_w > -1$, if $1/2 < p \le 1$, and $a_w < p-1,\ b_w > p-2$, if $p > 1$.

Then there exists a positive constant $c_{p,w}$ such that for every Blaschke product B we have

$$\|B'\|_{p,w}^p \le c_{p,w} \sum_{B(z)=0} (1-|z|)^{2-p}\, w(|z|)\,.$$

If $1/2 < p \le 1$ and $a_w = 2p-2,\ b_w > -1$, or if $p > 1,\ a_w = p-1,\ b_w > p-2$, and w also satisfies the condition
(*) there exists α such that $0 < \alpha < 1$ and the function

$$\frac{w(r)}{(1-r)^{a_w}} \log^\alpha \frac{1}{1-r}$$

is increasing for $r > r_0$,
then there exists $c_{p,w} > 0$ such that

$$\|B'\|_{p,w}^p \le c_{p,w} \times$$

$$\times \sum_{B(z)=0} (1-|z|)^{2-p}\, w(|z|) \log \frac{1}{1-|z|}$$

for every Blaschke product B.

Proof. Assume first that $1/2 < p \leq 1$. It is easy to see (after a logarithmic differentiation of the Blaschke product) that

$$|B'(\zeta)| \leq \sum_{B(z)=0} \frac{1-|z|^2}{|1-\overline{z}\zeta|^2}\,.$$

For $p \leq 1$ this implies

$$\begin{aligned}\int_{\mathbb{D}} |B'|^p\, w\, dA \quad &\leq \quad \sum_{B(z)=0} (1-|z|^2)^p \times \\ &\times \quad \int_{\mathbb{D}} \frac{w(\zeta)}{|1-\overline{z}\zeta|^{2p}}\, dA(z)\,.\end{aligned}$$

Let w be a normal weight and let $m \in \mathbb{R}$. For $\lambda \in \mathbb{D}$,$m > -1$, $a_w < m$, and $b_w > -1$, it can be checked that

$$\int_{\mathbb{D}} \frac{w(|z|)}{|1-\overline{\lambda}z|^{m+2}} dA(z) \asymp w(|\lambda|)(1-|\lambda|)^{-m}\,.$$

Take $m = 2p-2$ and the first part of the statement follows.

The other inequality is similar, using an analogous property of an integral involving the logarithm as well.

Theorem. Suppose that the zero set of the Blaschke product B is separated with separation constant $\delta > 0$. Suppose that either:
(a) $a_w < 2p-2$, $b_w > -1$, if $1/2 < p \le 1$, or
(b) $a_w < p-2$, $b_w > -1$, when $p > 1$.
Then, in both cases, there exists a positive constant $c_{p,w,\delta}$ such that

$$\sum_{B(z)=0}^{\infty} (1-|z|)^{2-p}\, w(|z|) \le c_{p,w,\delta} \|B'\|_{p,w}^p \,.$$

If either $1/2 < p \le 1$ and $a_w = 2p-2$, $b_w > -1$, or $p > 1$, $a_w = p-1$, $b_w > p-2$, and w also satisfies the condition (*) mentioned earlier, then there exists a positive constant $c_{p,w,\delta}$ such that

$$\sum_{B(z)=0}^{\infty} (1-|z|)^{2-p}\, w(|z|) \le c_{p,w,\delta} \times$$
$$\times \int_{\mathbb{D}} |B'(z)|^p w(|z|) \log \frac{1}{1-|z|} dA(z)\,.$$