Derivatives of Blaschke products and Bergman spaces with normal weights

(Joint work with A. Aleman)

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Blaschke products

\[ \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} : \text{unit disk.} \]

\{a_n\}: a sequence of points in \( \mathbb{D} \).

**Blaschke condition**: \( \sum_{n=1}^{\infty} (1 - |a_n|) < \infty \).

The corresponding **Blaschke product** \( B \) is defined as

\[
B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}.
\]

\( B \) is analytic in \( \mathbb{D} \), bounded by one, and has radial limits of modulus one almost everywhere on the unit circle.

Blaschke products are fundamental in the theory of Hardy spaces (being isometric zero-divisors).

**Question.** Study the behavior of \( B' \), *e.g.*, its membership in different function spaces.
Some contributors

Rudin, Piranian: initial examples (50's, 60's).

Ahern, Clark, Cohn, Protas, Kim (70's, 80's): systematic study of the membership of $B'$ in Hardy and Bergman spaces of the disk. Conditions on zeros.

Kutbi, Fricain, Mashreghi (2000-2010): recent contributions to the study of integral means of the derivative (on circles centered at the origin).


Bergman spaces

\[ z = re^{i\theta} \in \mathbb{D}, \quad dA(z) = \frac{1}{\pi} r \, dr \, d\theta. \]

For \( 0 < p < \infty \), we say that \( f \in L^p_a \) (the Bergman space) if

\[ \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty. \]

\( p > q \) implies \( L^p_a \subset L^q_a \).

Schwarz-Pick Lemma:

\[ |B'(z)| \leq \frac{1 - |B(z)|^2}{1 - |z|^2} \]

easily implies that \( B' \in \cap_{0 < p < 1} L^p_a \).

Rudin (1955): there exists \( B \) whose derivative is not in \( L^1_a \).

Piranian (1968): an explicit example.
Normal weights

Weight: a positive measurable function $w : \mathbb{D} \rightarrow [0, +\infty)$, usually integrable.

A weight is called radial if $w(z) = w(|z|)$ for all $z$ in $\mathbb{D}$.

A radial weight is normal (Shields, Williams) if there exist $a, b \in \mathbb{R}$ and $r_0 \in (0, 1)$ such that

$$\frac{w(r)}{(1 - r)^a} \nearrow \infty \quad \text{when} \quad r > r_a,$$

$$\frac{w(r)}{(1 - r)^b} \searrow 0 \quad \text{when} \quad r > r_b.$$

Let $a_w = \inf a$ and $b_w = \sup b$. Then $a_w \geq b_w$.

Example. Let $-1 < \alpha, \beta < \infty$, and consider

$$w(r) = (1 - r)^\alpha \log^\beta \frac{1}{1 - r}.$$

Then $a_w = b_w = \alpha$. 
Weighted Bergman spaces

We consider only integrable normal weight $w$. For example, this is guaranteed if we assume that $b_w > -1$.

Define the **weighted Bergman space** $L^p_\alpha(w)$, $p > 0$, to consists of all analytic functions $f$ in $\mathbb{D}$ such that

$$
\| f \|_{p,w}^p = \int_{\mathbb{D}} |f|^p w dA < \infty.
$$

From now on, assume: the weight $w$ is both normal and integrable in $[0, 1)$.

Such spaces generalize the standard Bergman spaces with

$$
w(z) = (\alpha + 1)(1 - r^2)^\alpha, \quad -1 < \alpha < \infty.
$$

For any normal and integrable weight it can be shown that $f \in L^p_\alpha(w)$ implies

$$
|f(z)| \leq C(w(|z|)^{-1/p}(1 - |z|^2)^{-2/p})\| f \|_{p,w},
$$

for some constant $C > 0$ and all $z \in \mathbb{D}$.
When $b_w > -1$, this implies
\[
\sup_{\|f\|_{p,w} \leq 1} |f(z)| \asymp (w(|z|))^{-1/p}(1 - |z|^2)^{-2/p},
\]
as $|z| \to 1^-$. 

**Interpolating sequences**

A sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{D}$ is called an interpolating sequence for $L_{a,w}^p$ if for every $(a_n) \in \ell^p$ there exists $f \in L_{a,w}^p$ such that
\[
f(z_n)w(|z_n|)^{1/p}(1 - |z_n|^2)^{2/p} = a_n,
\]
for all $n$.

If $p = 2$, it is well known that interpolating sequences for Hardy spaces are interpolating for $L_{a,w}^2$ as well. This follows from a well-known result that the only pointwise multipliers from $L_{a,w}^2$ into itself are the $H^\infty$ functions and from certain interpolation results.

This can be extended for the other values of $p > 1$ but requires different techniques.
**Theorem.** If $bw > -1$, then every interpolating sequence for $H^\infty$ is interpolating for $L^p_a(w), \ p > 1$.

**Proof.** If $\frac{1}{p} + \frac{1}{q} = 1$, let $\gamma > \frac{aw}{p} - \frac{1}{q}$.

Can show that

$$\left( \int_1^r w(t)dt \right)^{p/q} \left( \int_1^1 (1 - t)^q w^{-q/p}(t)dt \right) \leq C(1 - r)^{p\gamma + p}.$$  

This implies that $w(|z|)(1 - |z|)^{-\gamma}$ belongs to the Békollé class $B_p(\gamma)$.

Equivalently, $v(z) = w^{-q/p}(z)(1 - |z|)(q-1)^\gamma$ belongs to $B_q(\gamma)$.

By a result of Békollé (1982) it follows that the sublinear operator $P_\gamma$ defined by

$$P_\gamma f(z) = \int_D \frac{(1 - |\zeta|)^\gamma}{|1 - \bar{\zeta}z|^{\gamma+2}}|f(\zeta)|dA(\zeta)$$

is bounded from $L^q((1 - |z|)^q w^{-q/p}dA)$ into itself.
Luecking’s duality theorem (1985) identifies the dual of \( L^p_a(w) \) with \( L^q_a(v) \) via a standard weighted Bergman pairing. More precisely, it says that the norm of a function \( f \in L^p_a(w) \) is equivalent to

\[
\sup \left\{ \left| \int_D f(z)g(z)(1 - |z|^2)\gamma dA \right| : g \in L^q_a(w), \|g\|_{q,v} \leq 1 \right\}.
\]

Let \((z_n)_{n=1}^\infty\) be an interpolating sequence for \( H^\infty \), let \( B \) be the Blaschke product with zeros \( z_1, z_2, \ldots \) and let \( a = (a_n) \in \ell^p \).

We now exhibit an explicit solution \( f_a \) of the interpolation problem defined above:

\[
f_a(z) = \sum_n a_n w(|z_n|)^{-1/p} (1 - |z_n|^2)^{-2/p} \times
\]

\[
(1 - |z_n|^2)\gamma^1 B(z) \frac{1}{(1 - \bar{z}_n z)^\gamma + 1 (z - z_n) B'(z_n)}.
\]

In order to see this, it suffices to show that

\[
\|f_a\|_{p,w} \leq C \|a\|_{\ell^p},
\]

for some constant \( C \) and all sequences \( a \) in \( \ell^p \) with finitely many nonzero terms.
Denote by $B_n$ the Blaschke product with zeros $(z_k)_{k\neq n}$, taking multiplicities into account. Recall that the values $|B_n(z_n)|$ are bounded away from zero by the separation assumption. By the duality relation,

\[
\|f_a\|_{p,w} \leq \sup_{\|g\|_{q,v} \leq 1} \sum_n \frac{|a_n|}{|B_n(z_n)|} w^{-\frac{1}{p}}(|z_n|)(1 - |z_n|)^{\gamma + \frac{2}{q}} \times \\
\int_{\mathbb{D}} \frac{|g(z)||B_n(z)|(1 - |z|)^\gamma}{|1 - z_n z|^{\gamma + 2}} dA(z) \\
\leq C \sup_{\|g\|_{q,v} \leq 1} \sum_n |a_n| w^{-\frac{1}{p}}(|z_n|)(1 - |z_n|)^{\gamma + \frac{2}{q}} \times \\
P_{\gamma} g(z_n) \\
\leq C \|a\|_{\ell^p} \sup_{\|g\|_{q,v} \leq 1} \\
\left( \sum_n w^{-\frac{q}{p}}(|z_n|)(1 - |z_n|)^{q\gamma + 2} P_{\gamma}^q g(z_n) \right)^\frac{1}{q}.
\]
Since \((z_n)_{n=1}^\infty\) is uniformly separated, there is a fixed \(R \in (0,1)\) such that the disks
\[
\Delta_n = \{\zeta \in \mathbb{D} : |\zeta - z_n| < R(1 - |z_n|)\},
\]
are pairwise disjoint.

One can also see that \(P_\gamma g\) is “almost constant” on these disks, i.e., there exists \(c_0 > 0\) independent of \(g\) such that
\[
c_0^{-1} P_\gamma g(z) \leq P_\gamma g(\zeta) \leq c_0 P_\gamma g(z), \quad z, \zeta \in \Delta_n.
\]

Normal weights also behave nicely on such disks, so all together this yields
\[
\sup_{\|g\|_{q,v} \leq 1} \sum_n w^{-q/p}(|z_n|)(1 - |z_n|)^{q\gamma + 2} \times
\]
\[
\times P_\gamma^q g(z_n)
\]
\[
\leq C \sup_{\|g\|_{q,v} \leq 1} \sum_n w^{-q/p}(|z_n|) \times
\]
\[
\times (1 - |z_n|)^{q\gamma} \int_{\Delta_n} P_\gamma^q g(z)dA(z)
\]
\[
\leq C \sup_{\|g\|_{q,v} \leq 1} \sum_n \int_{\Delta_n} P_\gamma^q g(z) w^{-q/p}(z) \times
\]
\[
\times (1 - |z|)^{q\gamma} dA(z)
\]
\[ \leq \sup_{\|g\|_{q,v} \leq 1} \int_{\bigcup_n \Delta_n} P_q g(z) w^{-\frac{q}{p}}(z) (1 - |z|)^{\gamma} dA(z). \]

and the result follows.

**Integrability of \( B' \)**

For a nonconstant analytic function \( f \) in \( \mathbb{D} \), a point \( \zeta \) in \( \mathbb{D} \), and \( u : \mathbb{D} \to [0, +\infty) \), use the abbreviated notation

\[ \sum_{f(z) = \zeta} u(z) \]

to denote the summation over the set \( f^{-1}(\{\zeta\}) \), taking multiplicities into account.

Well-known:

\[ \|B'\|_1 \leq c \sum_{B(z) = 0} (1 - |z|) \log \frac{1}{1 - |z|}, \]

We complement this (and other results).
**Theorem** (H.O. Kim). Suppose that a Blaschke product \( B \) satisfies
\[
\sum_{B(z)=0} \infty (1 - |z|)^{2-p} < \infty, \quad 1 < p < 2.
\]
Then \( B' \in A^p \).

**Theorem** (Girela-Peláez-V.) If \( B \) is an interpolating Blaschke product then
\[
\int_{\mathbb{D}} |B'(z)|^p dA(z) \geq C \sum_{B(z)=0} (1 - |z|)^{2-p}.
\]
In particular, if the series on the right diverges, then \( B' \not\in A^p \).

A sequence \( \{a_n\}_n \) is said to be *separated* (with constant of separation \( \delta \)) if
\[
\inf_{k \neq n} \left| \frac{a_k - a_n}{1 - \overline{a_k}a_n} \right| = \delta > 0.
\]
Theorem. Let $a_w < 2p - 2$, $b_w > -1$, if $1/2 < p \leq 1$, and $a_w < p - 1$, $b_w > p - 2$, if $p > 1$.

Then there exists a positive constant $c_{p,w}$ such that for every Blaschke product $B$ we have

$$\|B'\|_{p,w}^p \leq c_{p,w} \sum_{B(z) = 0} (1 - |z|)^{2-p} w(|z|).$$

If $1/2 < p \leq 1$ and $a_w = 2p - 2$, $b_w > -1$, or if $p > 1$, $a_w = p - 1$, $b_w > p - 2$, and $w$ also satisfies the condition

(*) there exists $\alpha$ such that $0 < \alpha < 1$ and the function

$$\frac{w(r)}{(1 - r)^{a_w}} \log^\alpha \frac{1}{1 - r}$$

is increasing for $r > r_0$,

then there exists $c_{p,w} > 0$ such that

$$\|B'\|_{p,w}^p \leq c_{p,w} \times$$

$$\times \sum_{B(z) = 0} (1 - |z|)^{2-p} w(|z|) \log \frac{1}{1 - |z|}$$

for every Blaschke product $B$.  

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Proof. Assume first that $1/2 < p \leq 1$. It is easy to see (after a logarithmic differenti-
ation of the Blaschke product) that

$$|B'(\zeta)| \leq \sum_{B(z)=0} \frac{1 - |z|^2}{|1 - \bar{z}\zeta|^2}.$$  

For $p \leq 1$ this implies

$$\int_D |B'|^p w \, dA \leq \sum_{B(z)=0} (1 - |z|^2)^p \times \int_D \frac{w(\zeta)}{|1 - \bar{z}\zeta|^{2p}} \, dA(z).$$

Let $w$ be a normal weight and let $m \in \mathbb{R}$. For $\lambda \in \mathbb{D}, m > -1$, $a_w < m$, and $b_w > -1$, it can be checked that

$$\int_D \frac{w(|z|)}{|1 - \bar{\lambda}z|^{m+2}} \, dA(z) \asymp w(|\lambda|)(1 - |\lambda|)^{-m}.$$  

Take $m = 2p - 2$ and the first part of the statement follows.

The other inequality is similar, using an analogous property of an integral involving the logarithm as well.
**Theorem.** Suppose that the zero set of the Blaschke product $B$ is separated with separation constant $\delta > 0$. Suppose that either:
(a) $a_w < 2p - 2$, $b_w > -1$, if $1/2 < p \leq 1$, or
(b) $a_w < p - 2$, $b_w > -1$, when $p > 1$.
Then, in both cases, there exists a positive constant $c_{p,w,\delta}$ such that

$$\sum_{B(z)=0} \infty (1 - |z|)^{2-p} w(|z|) \leq c_{p,w,\delta} \|B'\|^p_{p,w}.$$ 

If either $1/2 < p \leq 1$ and $a_w = 2p - 2$, $b_w > -1$, or $p > 1$, $a_w = p - 1$, $b_w > p - 2$, and $w$ also satisfies the condition (*) mentioned earlier, then there exists a positive constant $c_{p,w,\delta}$ such that

$$\sum_{B(z)=0} \infty (1 - |z|)^{2-p} w(|z|) \leq c_{p,w,\delta} \times \int_{D} |B'(z)|^p w(|z|) \log \frac{1}{1 - |z|} dA(z).$$