Loewner theory in annulus

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Radial Loewner Theory: the beginning.

Class $\mathcal{S}$

By $\mathcal{S}$ we denote the class of all univalent (i.e. holomorphic and injective) functions $f : \mathbb{D} \to \mathbb{C}$ normalized by

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Bieberbach's Conject. (1916) - de Branges' Theorem (1984)

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Radial Loewner Theory: Kufarev and Pommerenke.

Important contributions by P.P. Kufarev and Ch. Pommerenke help Loewner's ideas to go beyond the initial goal.

The core of Loewner Theory resides in a 1-to-1 correspondence and interplay between the main notions:

- *Loewner chains*;
- *Evolution families*;
- *Herglotz vector fields*.

Radial Loewner Theory: Kufarev and Pommerenke.

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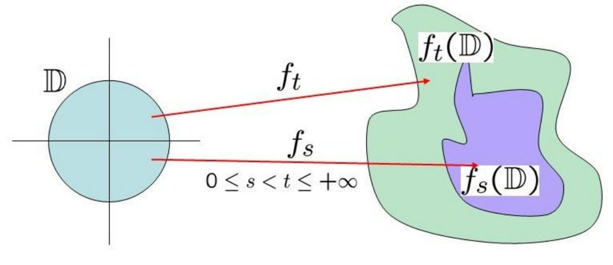

Radial Loewner Theory: Kufarev and Pommerenke

Theorem (Pommerenke, 1965)

For any $f \in \mathcal{S}$ there exists a radial Loewner chain (f_t) s.t. $f_0 = f$.

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For any radial Loewner chain (f_t) there exists a unique normalized Herglotz function $p(z,t)$ s.t. the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z}p(z,t),$$

holds.

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A *normalized Herglotz function* is $p : \mathbb{D} \times [0, +\infty)$ s. t.:

HF1 $z \mapsto p(z,t)$ is holomorphic for all t;

HF2 $t \mapsto p(z,t)$ is measurable for all z;

HF3 $\operatorname{Re} p(z,t) > 0$, $z \in \mathbb{D}$, $t \geq 0$, and $p(0,t) = 1$ for a.e. $t \geq 0$.

Radial Loewner Theory: Kufarev and Pommerenke.

The converse of the above theorem is true but, before stating it, we must recall:

Definition

The *radial evolution family* $(\varphi_{s,t})$ of a Loewner chain (f_t) is given by

$$\varphi_{s,t} = f_t^{-1} \circ f_s, \qquad 0 \leq s \leq t.$$

Proposition

A biparametric family of holomorphic functions $\varphi_{s,t} : \mathbb{D} \to \mathbb{D}$ *is a evolution family if and only if*

- EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ *for all* $s \geq 0$;
- EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ *whenever* $0 \leq s \leq u \leq t$;
- EF3 $\varphi_{s,t}(0) = 0$ *and* $\varphi'_{s,t}(0) = e^{s-t}$.

Radial Loewner Theory: Kufarev and Pommerenke

Theorem (Pommerenke)

For each normalized Herglotz function $p(z,t)$ there exists a unique Loewner chain (f_t) solving the Loewner PDE

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given by

$$f_s = \lim_{t\to+\infty} e^t \varphi_{s,t}(z), \quad s \geq 0,$$

where, given $s \geq 0$ and $z \in \mathbb{D}$, $\varphi_{s,t}(z) := w(t)$, being w the unique solution to the Loewner ODE

$$\begin{aligned} \frac{dw}{dt} &= -wp(w,t), \quad t \geq s, \\ w(s) &= z. \end{aligned}$$

Scheme

$$
\begin{array}{ccc}
\boxed{\text{Loewner chains } (f_t)} & \xleftrightarrow{\varphi_{s,t}=f_t^{-1}\circ f_s} & \boxed{\text{Evolution families } (\varphi_{s,t})} \\
\updownarrow & & \updownarrow \\
 & \boxed{\text{Herglotz vector fields } G(w,t)} &
\end{array}
$$

Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -G(z,t)\frac{\partial f_t(z)}{\partial z}$$

Loewner ODE

$$\frac{dw}{dt} = G(w,t),\ w(s) = z$$

$$\varphi_{s,t}(z) = w(t)$$

where $G(w,t) = -wp(w,t)$.

This scheme is important in applications.
For example: Bieberbach's conjecture.

Radial Loewner Theory: Kufarev and Pommerenke

In the sixties and seventies, using these results there was a number of applications (see the books by Pommerenke (1975) and Duren (1983)):

- univalence criteria,
- optimization problems (including partial results of the Bieberbach's conjecture),
- control theory,
- ...

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Also, there are connections with Fluid Mechanics because they can be used to model Hele-Shaw flows (see the book by Gustafsson and Vasil'ev (2006)).

Objective.

Problem:

To construct a Loewner theory in the doubly-connected setting.

We should introduce three notions:

Loewner chains, evolution families, and suitable vector fields

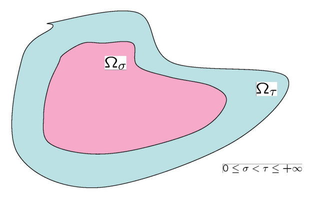
and analyze the relationships between them.

A geometric point of view: evolution of domains.

Any *continuously* increasing family of simply domains in the complex plane can be described by a Loewner chain.

That is: Let Ω_σ, $0 \leq \sigma < \infty$, be a family of simply connected domains s. t.

1. $0 \in \Omega_\sigma \subsetneq \Omega_\tau$ para $0 \leq \sigma < \tau$;
2. $\cup_{\tau<\sigma}\Omega_\tau = \Omega_\sigma$ and $\cup_\tau \Omega_\tau = \mathbb{C}$.



Then there is a Loewner chain (f_t) and a continuous and bijective function $\lambda : [0, +\infty) \to [0, +\infty)$ s. t. $f_t(\mathbb{D}) = \Omega_{\lambda(t)}$.

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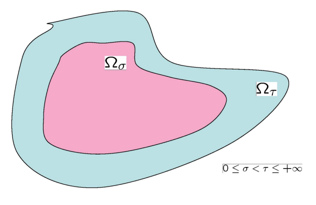

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Is there any similar construction for multiply-connected domains?

A geometric point of view: evolution of domains.

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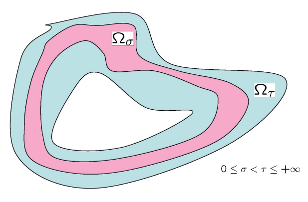
Let Ω_σ, $0 \leq \sigma < \infty$, be a family of doubly connected domains such that

1. $\Omega_\sigma \subsetneq \Omega_\tau$ para $0 \leq \sigma < \tau$;
2. $\cup_{\tau<\sigma}\Omega_\tau = \Omega_\sigma$.

Is there a "nice" family of univalent functions in an annulus $f_t : \mathbb{A} \to \mathbb{C}$ and a continuous and bijective function $\lambda : [0, +\infty) \to [0, +\infty)$ s. t. $f_t(\mathbb{A}) = \Omega_{\lambda(t)}$?



$\mathbb{A} := \mathbb{A}(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ where $0 \leq r < R \leq +\infty$.

Towards the definition of Loewner chain.

Let Ω_σ, $0 \leq \sigma < \infty$, be doubly connected domains s.t.

$$\Omega_\sigma \subsetneq \Omega_\tau \text{ if } 0 \leq \sigma < \tau \text{ and } \cup_{\tau<\sigma} \Omega_\tau = \Omega_\sigma.$$

Assume there is a "nice" family of univalent functions in an annulus $f_t : \mathbb{A} \to \mathbb{C}$ and a continuous and bijective function $\lambda : [0, +\infty) \to [0, +\infty)$ such that $f_t(\mathbb{A}) = \Omega_{\lambda(t)}$.

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"nice" should imply that $f_t \to f_s$ as $t \to s$ uniformly on compact sets.

Then $\varphi_{s,t} := f_t^{-1} \circ f_s : \mathbb{A} \to \mathbb{A}$ goes to $\mathrm{id}_{\mathbb{A}}$ as t goes to s.

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Connected component of $\mathrm{Hol}(D, D)$ containing id_D for any n-connected domain D, $n > 1$, is either $\{\mathrm{id}_D\}$ (when $n > 2$), or the family of all rotations (if $n = 2$).

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Thus, $\varphi_{s,t}$ is either the identity or a rotation.
In this case, $f_s(\mathbb{A}) = f_t \circ \varphi_{s,t}(\mathbb{A}) = f_t(\mathbb{A})$ and, in particular,

$$\Omega_\sigma = \Omega_\tau .$$

Towards the definition of Loewner chain.

With this approach, there is no evolution!

An idea:
To solve that problem, instead of a static reference domain, we can consider a family of reference domains (D_t), with Loewner chains being

$$f_t : D_t \to \mathbb{C}.$$

Towards the definition of Loewner chain.

For any $0 \leq r < 1$, we define $\mathbb{A}_r = \{z : r < |z| < 1\}$.

Definition

Let $r : [0, +\infty) \to [0, 1)$ be decreasing and locally absolutely continuous.

We say that $(\mathbb{A}_{r(t)})$ is a canonical domain system if

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If $r(t) > 0$ for all t, we say that $(\mathbb{A}_{r(t)})$ is *non-degenerate*.
If $r(t) = 0$ for all t, we say that $(\mathbb{A}_{r(t)})$ is *degenerate*.
Otherwise, we say that $(\mathbb{A}_{r(t)})$ is of *mixed type*.

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- LC3 ???

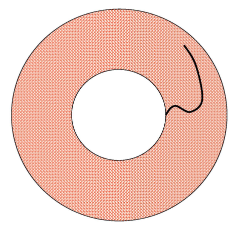
Towards the definition of Loewner chain.

In the fifties, there were several authors trying to tackle this problem:

- Komatu, 1943, 1950;
- Goluzin, 1951;
- Li En Pir, 1953;
- Lebedev, 1955;
- Kuvaev, Kufarev, 1955.

They developed a Loewner theory in the very particular case of

$$f_t(\mathbb{A}_{r(t)}) = \text{Annulus minus a slit}$$



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$$|f_s(z) - f_t(z)| \leq C(t - s)$$

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- F. Bracci, M.D. C., and S. Díaz-Madrigal, *Evolution Families and the Loewner Equation I: the unit disk*, to appear in J. Reine Angew. Math. Available on ArXiv 0807.1594

- M.D. C., S. Díaz-Madrigal, and P. Gumenyuk, *Loewner chains in the unit disk*. Revista Matemática Iberoamericana **26** (2010), 975–1012.

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- **LC3** for any compact interval $I = [S, T] \subset [0, +\infty)$ and any compact set $K \subset \mathbb{A}_{r(S)}$ there is a constant C such that
$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,I}(\xi)\, d\xi \quad \text{where } k_{K,I} \in L^d(I, [0, \infty))$$
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Evolution family: definition.

Now it is not so difficult to guess the notion of EF:

Definition

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- EF3 for any compact interval $I = [S, T] \subset [0, +\infty)$ and any $z \in \mathbb{A}_{r(S)}$ there is a constant C such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq C(t - u)$$

for all $S \leq s \leq u \leq t \leq T$.

Evolution family: definition.

Now it is not so difficult to guess the notion of EF:

Definition

Let $(\mathbb{A}_{r(t)})$ be a canonical domain system.
A family of holomorphic functions $\varphi_{s,t} : \mathbb{A}_{r(s)} \to \mathbb{A}_{r(t)}$, $0 \le s \le t$, is an *evolution family*, if

EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{A}_{r(s)}}$;

EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \le s \le u \le t$;

EF3 for any compact interval $I = [S, T] \subset [0, +\infty)$ and any $z \in \mathbb{A}_{r(S)}$ there is a constant C such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le C(t - u)$$

for all $S \le s \le u \le t \le T$.

Theorem

$\varphi_{s,t}$ is univalent in $\mathbb{A}_{r(s)}$.

Loewner chains and evolution families

Theorem

Let (f_t) be a Loewner chain over $(\mathbb{A}_{r(t)})$. If we define

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \qquad 0 \leq s \leq t < \infty, \tag{2.1}$$

then $(\varphi_{s,t})$ is an evolution family over $(\mathbb{A}_{r(t)})$.

Loewner chains and evolution families

Theorem

Let $(\varphi_{s,t})$ be an evolution fam. over $(\mathbb{A}_{r(t)})$. Let $r_\infty := \lim_{t\to+\infty} r(t)$. Then there exists a Loewner chain (f_t) over $(\mathbb{A}_{r(t)})$ such that

1. $f_s = f_t \circ \varphi_{s,t}$ *for all* $0 \le s \le t < +\infty$;
2. $I(f_t \circ \gamma) = I(\gamma)$ *for any closed curve* $\gamma \subset A_{r(t)}$ *and any* $t \ge 0$, *where* $I(\gamma)$ *denote the index of the origin w.r.t. a closed curve in* $\mathbb{C}^*$;
3. *If* $0 < r_\infty < 1$, *then* $\cup_{t\in[0,+\infty)} f_t(A_{r(t)}) = \mathbb{A}_{r_\infty}$;
4. *If* $r_\infty = 0$, *then* $\cup_{t\in[0,+\infty)} f_t(\mathbb{A}_{r(t)})$ *is either* $\mathbb{D}^*$, $\mathbb{C}\setminus\overline{\mathbb{D}}$, *or* $\mathbb{C}^*$.

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If (g_t) is another Loewner chain over $(\mathbb{A}_{r(t)})$ such that $g_s = g_t \circ \varphi_{s,t}$, for all $0 \leq s \leq t$, then there is a biholomorphism

$$F : \cup_{t\in[0,+\infty)} f_t(\mathbb{A}_{r(t)}) \to \cup_{t\in[0,+\infty)} g_t(\mathbb{A}_{r(t)})$$

such that

$$g_t = F \circ f_t, \quad \textit{for all } t \geq 0.$$

What about the vector field?

Definition

Let $(\mathbb{A}_{r(t)})$ be a canonical domain system.
If $r(t) > 0$ for all t, we say that $(\mathbb{A}_{r(t)})$ is non-degenerate.
If $r(t) = 0$ for all t, we say that $(\mathbb{A}_{r(t)})$ is degenerate.
Otherwise, we say that $(\mathbb{A}_{r(t)})$ is of mixed type.

Degenerate case: $\mathbb{A}_{r(t)} = \mathbb{D} \setminus \{0\}$ for all t.

In this case, $\varphi_{s,t} : \mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$.

Defining $\varphi_{s,t}(0) = 0$, we get analytic functions in the unit disk and we are in the simply connected case.

In the rest of this talk, we will concentrate in the non-degenerate case.

What about the vector field?

To imagine the kind of vector field we are looking for, go back to $\mathbb{D}$:

Theorem (Pommerenke, 1965)

For any radial Loewner chain (f_t) **in the unit disk** *there exists a unique normalized Herglotz function* $p(z,t)$ *s.t. the Loewner PDE*

$$\frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z}p(z,t),$$

holds, where

HF1 $z \mapsto p(z,t)$ *is holomorphic for all* t*;*

HF2 $t \mapsto p(z,t)$ *is measurable for all* z*;*

HF3 $\operatorname{Re} p(z,t) > 0$, $z \in \mathbb{D}$, *and* $p(0,t) = 1$ *for a.e.* $t \geq 0$.

Notice that for each t there is a positive Borel measure μ^t on the unit circle $\partial\mathbb{D}$ such that

$$p(z,t) = \int_{\partial\mathbb{D}} \frac{z+\xi}{z-\xi}\,\mu^t(\xi), \quad (z \in \mathbb{D}).$$

What about the vector field?

The analogue of the Schwartz kernel $\mathcal{K}_0(z) := (1+z)/(1-z)$ for $\mathbb{A}_r := \{z : r < |z| < 1\}$, $r \in (0,1)$, is the *Villat kernel*:

$$\mathcal{K}_r(z) := \lim_{n\to+\infty} \sum_{\nu=-n}^{n} \frac{1+r^{2\nu}z}{1-r^{2\nu}z} = \frac{1+z}{1-z} + \sum_{\nu=1}^{+\infty} \left(\frac{1+r^{2\nu}z}{1-r^{2\nu}z} + \frac{1+z/r^{2\nu}}{1-z/r^{2\nu}} \right).$$

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This kernel plays the same role for the Function Theory in the annulus than the Schwartz kernel in the unit disk.
Let us mention that for any $f \in \mathrm{Hol}(\mathbb{A}_r, \mathbb{C})$, continuous in $\overline{\mathbb{A}_r}$,

$$f(z) = \int_{\partial\mathbb{D}} \mathcal{K}_r(z\xi^{-1}) \mathrm{Re}\, f(\xi) \frac{|d\xi|}{2\pi} + \int_{\partial\mathbb{D}} \left[\mathcal{K}_r(r\xi/z) - 1\right] \mathrm{Re}\, f(r\xi) \frac{|d\xi|}{2\pi} + i \int_{\partial\mathbb{D}} \mathrm{Im}\, f(\xi) \frac{|d\xi|}{2\pi}$$

for all $z \in \mathbb{A}_r$.

What about the vector field?

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Definition

Let $r \in (0,1)$. By *the class* $\mathcal{V}_r$ we will mean the collection of all $p \in \mathrm{Hol}(\mathbb{A}_r, \mathbb{C})$ having the following representation

$$p(z) = \int_{\partial\mathbb{D}} \mathcal{K}_r(z/\xi)\, d\mu_1(\xi) + \int_{\partial\mathbb{D}} \left[1 - \mathcal{K}_r(r\xi/z)\right] d\mu_2(\xi), \quad z \in \mathbb{A}_r,$$

where μ_1 and μ_2 are positive Borel measures on the unit circle $\partial\mathbb{D}$ subject to the condition $\mu_1(\partial\mathbb{D}) + \mu_2(\partial\mathbb{D}) = 1$.

Vector fields and evolution families

Theorem

For any evolution family $(\varphi_{s,t})$ *over* $(\mathbb{A}_{r(t)})$ *there exist an essentially unique semicomplete weak holomorphic vector field*

$$G(w,t) = w\big[iC(t) + r'(t)p(w,t)/r(t)\big]$$

for a.e. $t \geq 0$ *and all* $w \in \mathbb{A}_{r(t)}$*, where*

1. $t \mapsto p(w,t)$ *is measurable in* (t_0, ∞)*, where* $t_0 := \inf\{t : \ w \in \mathbb{A}_{r(t)}\}$*;*
2. *for each* $t \geq 0$ *the function* $p(\cdot\,, t)$ *belongs* $\mathcal{V}_{r(t)}$*;*
3. $C \in L^{\infty}_{\mathrm{loc}}([0,+\infty),\mathbb{R})$*.*

and a null-set $N \subset [0,+\infty)$ *such that, for all* $s \geq 0$*, the mapping* $[s,+\infty) \ni t \mapsto \varphi_{s,t} \in \mathrm{Hol}(\mathbb{A}_{r(s)},\mathbb{C})$ *is differentiable for all* $t \in [s,+\infty) \setminus N$ *and*

$$\frac{d\varphi_{s,t}}{dt} = G(\cdot,t) \circ \varphi_{s,t} \quad \textit{for all } t \in [s,+\infty) \setminus N.$$

Vector fields and evolution families

Conversely,

Theorem

Given any function

$$G(w,t) = w\big[iC(t) + r'(t)p(w,t)/r(t)\big]$$

for a.e. $t \geq 0$ *and all* $w \in \mathbb{A}_{r(t)}$, *where*

1. $t \mapsto p(w,t)$ *is measur. in* (t_0, ∞), *where* $t_0 := \inf\{t : \ w \in \mathbb{A}_{r(t)}\}$;
2. *for each* $t \geq 0$ *the function* $p(\cdot\,,t)$ *belongs* $\mathcal{V}_{r(t)}$;
3. $C \in L^\infty_{\mathrm{loc}}([0,+\infty),\mathbb{R})$

any $s \geq 0$ *and any* $w \in \mathbb{A}_{r(s)}$, *the IVP*

$$\dot{w} = G(w,t), \quad w(s) = z,$$

has a unique solution defined in $[s,+\infty)$ *such that the formula* $\varphi_{s,t}(z) := w(t)$ *is an evolution family over* $(\mathbb{A}_{r(t)})$.

PDE and Lowner chains

Corollary

Let (f_t) be a Loewner chain over $(\mathbb{A}_{r(t)})$. Then:

(i) *There exists a null-set $N \subset [0, +\infty)$ (not depending on z) such that for every $s \in [0, +\infty) \setminus N$ the function*

$$z \in \mathbb{A}_{r(s)} \mapsto \frac{\partial f_s(z)}{\partial s} := \lim_{h \to 0} \frac{f_{s+h}(z) - f_s(z)}{h} \in \mathbb{C}$$

is a well-defined holomorphic function on $\mathbb{A}_{r(s)}$.

(ii) *There exists an essentially unique weak holomorphic vector field $G(w,t) = w\left[iC(t) + r'(t)p(w,t)/r(t)\right]$ over $(\mathbb{A}_{r(t)})$ such that for a.e. $s \in [0, +\infty)$,*

$$\frac{\partial f_s(z)}{\partial s} = -G(z,s) f_s'(z) \quad \text{for all } z \in \mathbb{A}_{r(s)}.$$

Scheme

Loewner chains (f_t) $\xleftrightarrow{\varphi_{s,t}=f_t^{-1}\circ f_s}$ Evolution families $(\varphi_{s,t})$

Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -G(z,t)\frac{\partial f_t(z)}{\partial z}$$

Loewner ODE

$$\frac{dw}{dt} = G(w,t),\ w(s) = z$$

$$\varphi_{s,t}(z) = w(t)$$

Vector fields $G(w,t)$

where $G(w,t) = w\big[iC(t) + r'(t)p(w,t)/r(t)\big]$.

- M. D. C., S. Díaz-Madrigal, and P. Gumenyuk, *Loewner Theory in annulus I: evolution families and differential equations.* Preprint 2010. Availiable on arXiv:1011.4253

- M. D. C., S. Díaz-Madrigal, and P. Gumenyuk, *Loewner Theory in annulus II: Loewner chains.* Preprint 2011. Availiable on arXiv:1105.3187