Equality cases for condenser capacity inequalities under symmetrization.

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Part I

Definitions

Definition

A **condenser** *in* $\mathbb{R}^n$, $n \geq 2$, *is a pair* (D, K), *where* D *is an open subset of* $\mathbb{R}^n$ *and* K *is a nonempty compact subset of* D.

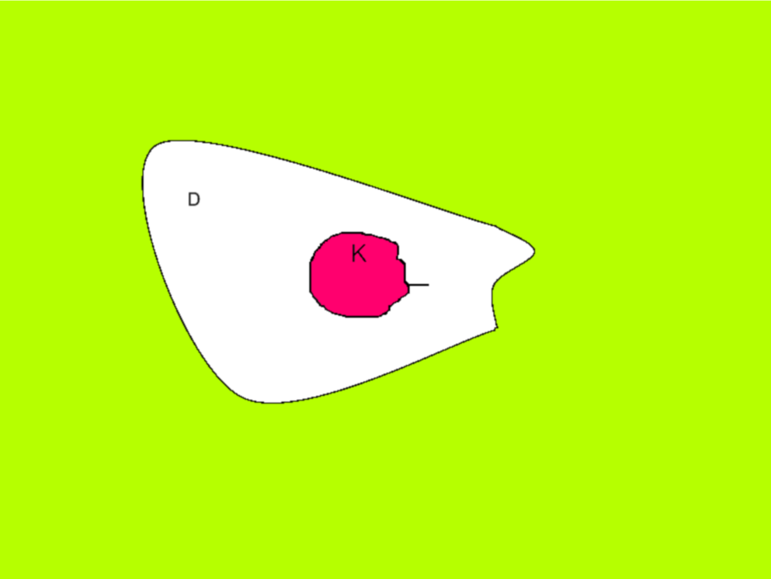
Definition

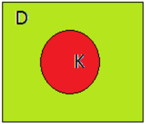
The **equilibrium potential** h *of the condenser* (D, K) *is the solution of the generalized Dirichlet problem on* $D \setminus K$ *with boundary values* 0 *on* ∂D *and* 1 *on* ∂K.

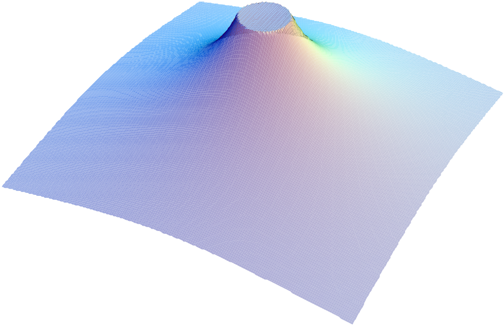
Definition

The **capacity** *of* (D, K) *is*

$$\mathrm{Cap}(D, K) = \int_{D \setminus K} |\nabla h|^2.$$


D
K


D
K



$C_2(E)$ is the logarithmic ($n = 2$) or Newtonian ($n \geq 3$) capacity of the Borel set $E \subset \mathbb{R}^n$.

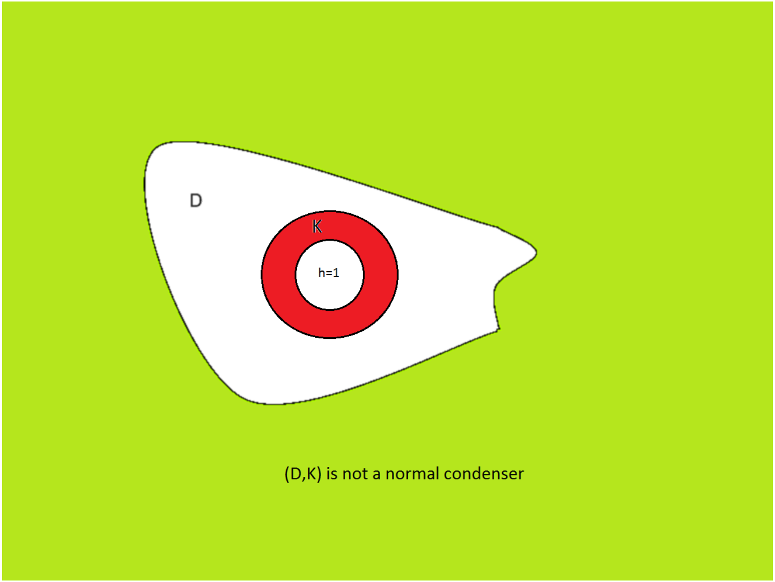
$$A \overset{n.e.}{=} B \iff C_2(A \setminus B) = C_2(B \setminus A) = 0.$$

A, B are **nearly everywhere equal**.

Definition

A condenser (D, K) *will be called* **normal** *if for every connected component* Ω *of* $D \setminus K$*, it is true that*

$$C_2(\partial D \cap \partial\Omega) > 0 \qquad \text{and} \qquad C_2(\partial K \cap \partial\Omega) > 0.$$


D
K
h=1
(D,K) is not a normal condenser

Polarization

Let H be an oriented $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ and let H^+ and H^- be the closed half-spaces into which H divides $\mathbb{R}^n$, with respect to the given orientation.

We denote by $R_H(\cdot)$ the reflection of a point or a subset of $\mathbb{R}^n$ in H.

Let E be any set in $\mathbb{R}^n$. We divide E into three disjoint sets as follows: The *symmetric part* of E is the set

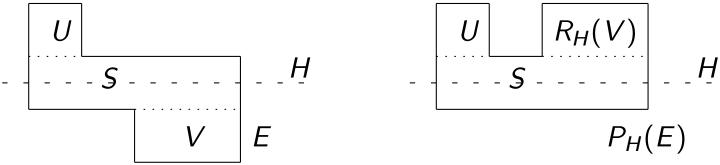
$$S_E = \{x \in E : R_H(x) \in E\},$$

the *upper non-symmetric part* of E is the set

$$U_E = \{x \in E \cap H^+ : R_H(x) \notin E\}$$

and the *lower non-symmetric part* of E is the set

$$V_E = \{x \in E \cap H^- : R_H(x) \notin E\}.$$

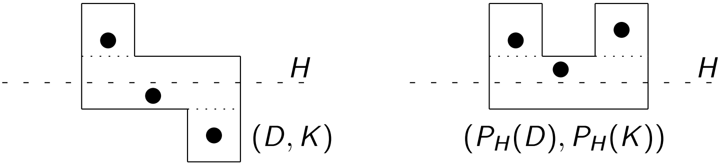


The polarization $P_H(E)$ of E with respect to H is the set

$$P_H(E) := S_E \cup U_E \cup R_H(V_E).$$

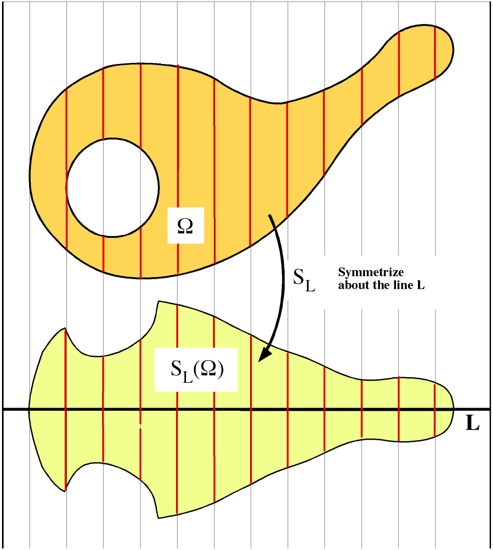
The polarization of an open (closed) set is open (closed).

The polarization of a condenser (D, K) is the condenser $(P_H(D), P_H(K))$.

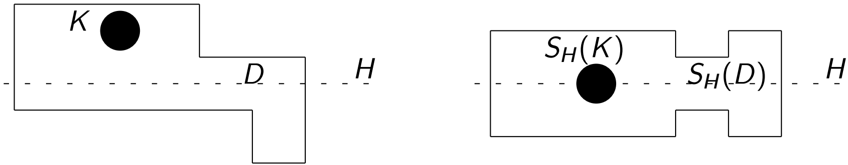


Steiner symmetrization

We define the Steiner symmetrization $S_H(A)$ of an open or compact set $A \subset \mathbb{R}^n$ with respect to the $(n-1)$-dimensional hyperplane H by determining its intersections with every line perpendicular to H.


Ω
S_L
Symmetrize
about the line L
S_L(Ω)
L

The Steiner symmetrization of a condenser (D, K) with respect to H is the condenser $(S_H(D), S_H(K))$.



- Polarization is a much simpler geometric transformation than Steiner symmetrization.
- An important property of polarization is that the Steiner symmetrization can be approximated by a sequence of suitable polarizations.
- V. N. Dubinin first introduced polarization of condensers and concidered the approach to symmetrization of condensers via polarization.

Part II

Condenser capacity inequalities

It is true that, polarization and Steiner symmetrization decreases condenser capacity; that is, for every condenser (D, K) and for every hyperplane H,

$$\mathrm{Cap}(P_H(D), P_H(K)) \leq \mathrm{Cap}(D, K) \tag{1}$$

and

$$\mathrm{Cap}(S_H(D), S_H(K)) \leq \mathrm{Cap}(D, K). \tag{2}$$

Inequality (2) proved by G. Pólya and G. Szegö (1951).
Inequality (1) proved by V. N. Dubinin (1985).

Equality statements for the inequalities (1) and (2) have been proved by J. A. Jenkins, V. N. Dubinin and V. A. Shlyk under connectivity assumptions or regularity conditions on the boundary of the condenser.

Proposition

Let (D, K) be a condenser such that $D \setminus K$ is connected. Then

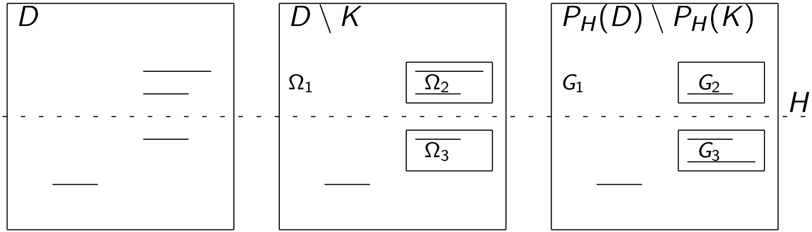
$$\mathrm{Cap}(P_H(D), P_H(K)) = \mathrm{Cap}(D, K)$$

if and only if either

$$P_H(D) \overset{n.e.}{=} D \text{ and } P_H(K) \overset{n.e.}{=} K$$

or

$$P_H(D) \overset{n.e.}{=} R_H(D) \text{ and } P_H(K) \overset{n.e.}{=} R_H(K).$$


D
$D \setminus K$
Ω_1
Ω_2
Ω_3
$P_H(D) \setminus P_H(K)$
G_1
G_2
G_3
H

Theorem (Equality statement for Steiner symmetrization)

Let (D, K) be a normal condenser, let $\{D_i\}$ be the connected components of D, $K_i = K \cap D_i$ and let H be a plane in $\mathbb{R}^n$. Then

$$\mathrm{Cap}(S_H(D), S_H(K)) = \mathrm{Cap}(D, K)$$

if and only if for every i there exists a plane H_i parallel to H such that

$$S_{H_i}(D_i) \overset{n.e.}{=} D_i, \qquad S_{H_i}(K_i) \overset{n.e.}{=} K_i$$

and $\Pi_H(D_i) \cap \Pi_H(D_j) = \varnothing$ for every $i \neq j$, where Π_H is orthogonal projection on H.

Suppose that $D \setminus K$ is connected.
Let Y be an oriented hyperplane which is parallel to H.
Observe that for any open or compact set A,

$$S_H(P_Y(A)) = S_H(A).$$

From the condenser capacity inequalities,

$$\begin{aligned}
\mathrm{Cap}(D,K) &\geq \mathrm{Cap}(P_Y(D),P_Y(K)) \\
&\geq \mathrm{Cap}(S_H(P_Y(D)),S_H(P_Y(K))) \\
&= \mathrm{Cap}(S_H(D),S_H(K)) \\
&= \mathrm{Cap}(D,K)
\end{aligned}$$

and

$$\mathrm{Cap}(D,K) = \mathrm{Cap}(P_Y(D),P_Y(K)).$$

Therefore, for every oriented hyperplane Y which is parallel to H,

$$P_Y(D) \overset{n.e.}{=} D \text{ and } P_Y(K) \overset{n.e.}{=} K$$

or

$$P_Y(D) \overset{n.e.}{=} R_Y(D) \text{ and } P_Y(K) \overset{n.e.}{=} R_Y(K).$$

The assertion follows from the following lemma about polarization and Steiner symmetrization.

Lemma

If for every oriented hyperplane Y which is parallel to a given hyperplane H,

$$P_Y(A) \overset{n.e.}{=} A \qquad or \qquad P_Y(A) \overset{n.e.}{=} R_Y(A),$$

then there exists a hyperplane Y_0 which is parallel to H, such that $S_{Y_0}(A) \overset{n.e.}{=} A$.

Thank you!

Theorem (Equality statement for polarization)

Let (D, K) be a normal condenser and let H be an oriented plane in $\mathbb{R}^n$. Then $\mathrm{Cap}(P_H(D), P_H(K)) = \mathrm{Cap}(D, K)$ if and only if there is a one-to-one and onto correspondence between the connected components Ω' of $P_H(D) \setminus P_H(K)$ and the connected components Ω of $D \setminus K$ such that either

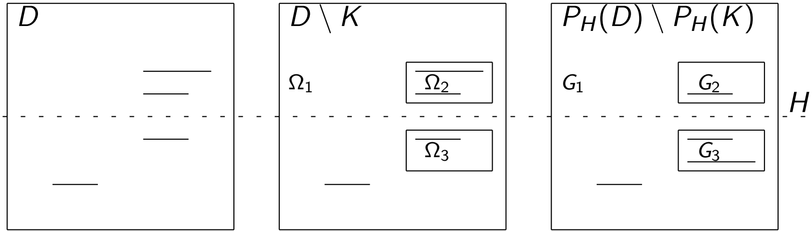
$$\Omega' \overset{n.e.}{=} \Omega, \qquad \partial\Omega' \cap \partial P_H(K) \overset{n.e.}{=} \partial\Omega \cap \partial K$$

$$\text{and} \qquad \partial\Omega' \cap \partial P_H(D) \overset{n.e.}{=} \partial\Omega \cap \partial D$$

or

$$\Omega' \overset{n.e.}{=} R_H(\Omega), \qquad \partial\Omega' \cap \partial P_H(K) \overset{n.e.}{=} R_H(\partial\Omega \cap \partial K)$$

$$\text{and} \qquad \partial\Omega' \cap \partial P_H(D) \overset{n.e.}{=} R_H(\partial\Omega \cap \partial D).$$


D
$D \setminus K$
Ω_1
Ω_2
Ω_3
$P_H(D) \setminus P_H(K)$
G_1
G_2
G_3
H

For the proof we consider the equilibrium potential h of (D, K) and the polarization of h which is the function

$$Ph(x) = \begin{cases} \min\{h(x), h(R_H(x))\}, & x \in H^-, \\ \max\{h(x), h(R_H(x))\}, & x \in H^+. \end{cases}$$

It is true that Ph is ACL on $P_H(D) \setminus P_H(K)$, it has boundary limits 0 on $\partial P_H(D)$ and 1 on $\partial P_H(K)$ for all boundary points which are regular for the Dirichlet problem and

$$\int_{P_H(D)\setminus P_H(K)} |\nabla Ph|^2 = \int_{D\setminus K} |\nabla h|^2.$$

Let u be the equilibrium potential of the condenser $(P_H(D), P_H(K))$.

From the extended Dirichlet principle

$$\begin{aligned}
\mathrm{Cap}(P_H(D), P_H(K)) &= \int_{P_H(D)\setminus P_H(K)} |\nabla u|^2 \\
&\leq \int_{P_H(D)\setminus P_H(K)} |\nabla Ph|^2 \\
&= \int_{D\setminus K} |\nabla h|^2 \\
&= \mathrm{Cap}(D, K) \\
&= \mathrm{Cap}(P_H(D), P_H(K))
\end{aligned}$$

and therefore

$$\mathrm{Cap}(P_H(D), P_H(K)) = \int_{P_H(D)\setminus P_H(K)} |\nabla u|^2 = \int_{P_H(D)\setminus P_H(K)} |\nabla Ph|^2.$$

From the equality statement of the extended Dirichlet principle we obtain that Ph is the equilibrium potential of the condenser $(P_H(D), P_H(K))$.

Next, from the boundary behavior of the equilibrium potential and the assumption that (D, K) is a normal condenser, follows the conclusion of the theorem.