Rate of growth of D-frequently hypercyclic functions

A. Bonilla

Departamento de Análisis Matemático
Universidad de La Laguna

Hypercyclic

Definition

A (linear and continuous) operator T in a topological vector space X is said to be hypercyclic if there exists a vector $x \in X$, also called hypercyclic, whose orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X.

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Examples:

T_a Birkhoff (1929)

D MacLane (1952/53)

Frequently hypercyclic

Definition (Bayart and Grivaux, 2006)

Let X be a topological vector space and $T : X \to X$ a operator. Then a vector $x \in X$ is called frequently hypercyclic for T if, for every non-empty open subset U of X,

$$\underline{dens}\{n \in \mathbb{N} : T^n x \in U\} > 0.$$

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The lower density of a subset A of $\mathbb{N}$ is defined by

$$\underline{\text{dens}}(A) = liminf_{N\to\infty} \frac{\#\{n \in A : n \leq N\}}{N},$$

where $\#$ denotes the cardinality of a set.

Remark

A vector $x \in X$ is frequently hypercyclic for an operator T in X if and only if for every non-empty open subset U of X, there is a strictly increasing sequence (n_k) of positive integers and some $C > 0$ such that

$$n_k \leq Ck \quad \text{and} \quad T^{n_k}x \in U \quad \text{para todo } \; k \in \mathbb{N}.$$

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In the known examples of FHC operators, in particular, D and T_a $FHC(T)$ is a dense set of first category.

Order of growth and D-frequently hypercyclic function

Given a entire function f is defined for $1 \leq p < \infty$,

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p}, r > 0.$$

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Theorem (K.-G. Grosse-Erdmann, 1990)

Let $1 \leq p \leq \infty$*.*
(a) For any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\varphi(r) \to \infty$ *as* $r \to \infty$ *there is a D-hypercyclic entire function f with*

$$M_p(f,r) \leq \varphi(r) \frac{e^r}{\sqrt{r}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

(b) There is no D-hypercyclic entire function f that satisfies

$$M_p(f,r) \leq C \frac{e^r}{\sqrt{r}} \quad \text{for } r > 0,$$

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- [B. and Grosse-Erdmann, 2006] $\forall \varepsilon > 0$ there exist entire functions D-frequently hypercyclic with

$$|f(z)| \leq Ce^{(1+\varepsilon)r} \text{ for } |z| = r.$$

Does there exist an entire function D-frequently hypercyclic f with

$$M_\infty(f,r) \leq \frac{e^r}{r^a} \quad \text{for } |z| = r \text{ sufficiently large?}$$

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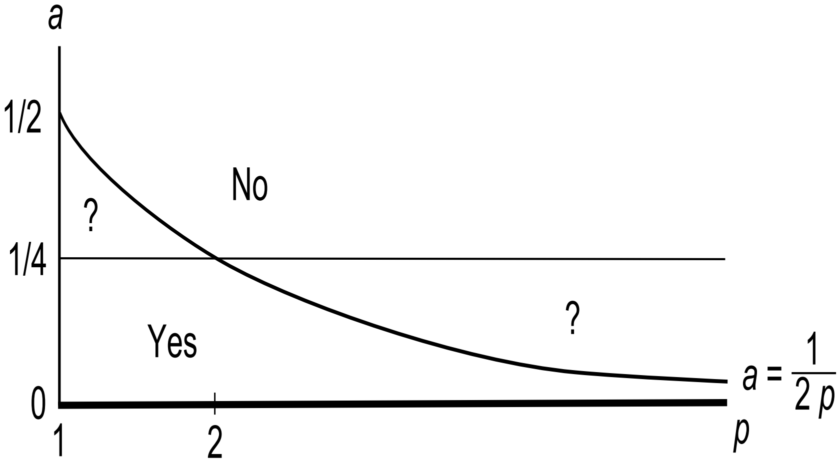

Theorem (Blasco-B-Grosse-Erdmann, PEMS 2009)

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\psi(r) \to 0$ as $r \to \infty$.
a)If $1 \leq p \leq 2$, there is no D-frequently hypercyclic entire function f that satisfies

$$M_p(f,r) \leq \psi(r)\frac{e^r}{r^{1/2p}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

b)If $2 < p \leq \infty$, there is no D-frequently hypercyclic entire function f that satisfies

$$M_p(f,r) \leq \psi(r)\frac{e^r}{r^{1/4}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

Lemma

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\psi(r) \to 0$ as $r \to \infty$, $1 < p \leq 2$ and f is an entire function that satisfies

$$M_p(f,r) \leq \psi(r)\frac{e^r}{r^{1/2p}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

Then

$$\frac{1}{m}\sum_{n=0}^{m} |f^{(n)}(0)|^q \to 0.$$

where q is the conjugate exponent of p.

Proof of theorem

For p=1, is consequence of Grosse-Erdmann's theorem. (There is no D-hypercyclic entire function f that satisfies

$$M_p(f,r) \leq C\frac{e^r}{\sqrt{r}} \quad \text{for } r > 0,$$

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$$\begin{aligned}\underline{\text{dens}}\{n \in \mathbb{N} : f^{(n)} \in U\} &= liminf_{m\to\infty}\frac{1}{m}\#\{n \leq m : |f^{(n)}(0)| > 1\} \\ &\leq liminf_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m}|f^{(n)}(0)|^q = 0,\end{aligned}$$

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Then

$$M_2(f,r) = \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \right)^{1/2} \leq \psi(r) \frac{e^r}{r^{1/4}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

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Hence, for big values of r,

$$\sum_{n=0}^{\infty} |f^{(n)}(0)|^2 \frac{r^{2n+1/2}e^{-2r}}{\psi(r)^2(n!)^2} \leq C$$

Fix $m \geq 1$. Using Stirling formula, we see that the function

$$r \mapsto \frac{r^{2n+1/2} e^{-2r}}{(n!)^2}$$

has its maximum at $n+1/4$ of order $1/\sqrt{n}$ and an inflection point at $n+1/4 \pm \sqrt{\frac{n}{2}+\frac{1}{8}}$. Hence, if $m \leq n < 2m$ then

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$$\int_m^{3m} \frac{r^{2n+1/2} e^{-2r}}{\psi(r)^2 (n!)^2} \geq C \frac{1}{\psi(m)^2} \frac{1}{\sqrt{n}} \sqrt{\frac{n}{2} + \frac{1}{8}} \geq C \frac{1}{\psi(m)^2}.$$

Integrating

$$\sum_{n=0}^{\infty} |f^{(n)}(0)|^2 \frac{r^{2n+1/2} e^{-2r}}{\psi(r)^2 (n!)^2} \leq C$$

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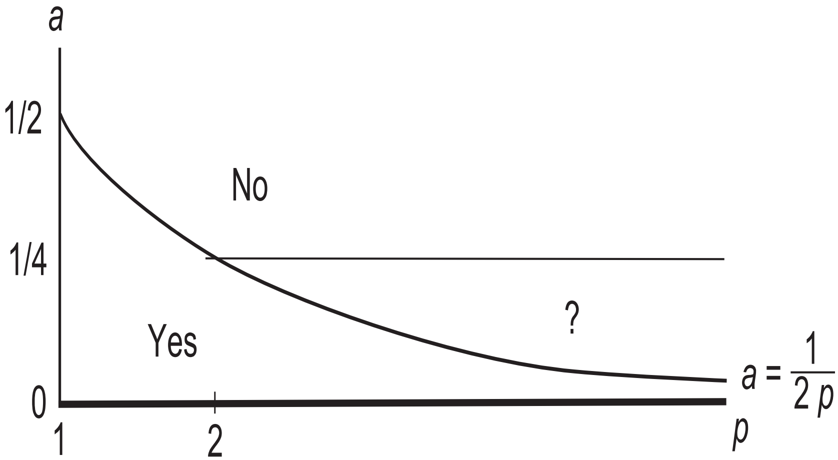
This implies

$$\frac{1}{m}\sum_{n=0}^{m} |f^{(n)}(0)|^2 \to 0.$$

If $1 \leq p < 2$, using Hausdorff-Young inequality

$$\Big(\sum_{n=0}^{\infty} |a_n|^q r^{qn}\Big)^{\frac{1}{q}} \leq M_p\Big(\sum_{n=0}^{\infty} a_n z^n, r\Big)$$

and the same ideas for $p = 2$.


a
1/2
1/4
0
No
?
Yes
$a = \frac{1}{2p}$
1
2
p

Theorem (Bonet-B., CAOT 2011)

For any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$. If $1 \leq p \leq \infty$, there is an entire function D-frequently hypercyclic f with

$$M_p(f, r) \leq \varphi(r) \frac{e^r}{r^{1/2p}} \text{ for } |z| = r \text{ sufficiently large.}$$

For $1 \leq p \leq \infty$ and a weight function v,

$$B_{p,\infty} = B_{p,\infty}(\mathbb{C}, v) := \{f \in H(\mathbb{C}) : \sup_{r>0} v(r) M_p(f, r) < \infty\}$$

and

$$B_{p,0} = B_{p,0}(\mathbb{C}, v) := \{f \in H(\mathbb{C}) : lim_{r \to \infty} v(r) M_p(f, r) = 0\}.$$

These spaces are Banach spaces with the norm

$$\|f\|_{p,v} := \sup_{r>0} v(r) M_p(f, r).$$

Remarks

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2.$B_{p,0}$ is separable.

3.The inclusion $B_{p,\infty} \subset H(\mathbb{C})$ is continuous.

Lemma

Let v be a weight function such that $\sup_{r>0} \dfrac{v(r)}{v(r+1)} < \infty$. *Then the differentiation operators* $D : B_{p,\infty} \to B_{p,\infty}$ *and* $D : B_{p,0} \to B_{p,0}$ *are continuous.*

Example: $v(r) = e^{-ar}, r > 0, a > 0$.

Theorem (Grivaux, 2011)

Let T be a bounded operator on a complex separable Banach space X. Suppose that, for any countable set $A \subset \mathbb{T}$, $\text{span}\{Ker(T - \lambda I) : \lambda \in \mathbb{T} \setminus A\}$ is dense in X. Then T is frequently hypercyclic.

Lemma

The following conditions are equivalent for a weight v and $1 \leq p < \infty$:

(i) $\{e^{\theta z} : |\theta| = 1\} \subset B_{p,0}$.

(ii) *There is* $\theta \in \mathbb{C}, |\theta| = 1$, *such that* $e^{\theta z} \in B_{p,0}$.

(iii) $lim_{r\to\infty} v(r) \dfrac{e^r}{r^{\frac{1}{2p}}} = 0$

Proof of the Lemma

Consider $f(z) = e^z, z \in \mathbb{C}$, and write $z = r(cost + isint)$. Now, we can apply the Laplace methods for integrals, for $r > 0$,

$$2\pi M_p(e^z, r)^p = \int_0^{2\pi} e^{rp\cos t} dt = \left(\frac{\pi}{2rp}\right)^{1/2} e^{rp} + e^{rp} O\left(\frac{1}{rp}\right).$$

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This yields, for a certain constant $c_p > 0$ depending only on p,

$$M_p(f, r) = c_p \frac{e^r}{r^{\frac{1}{2p}}} + e^r O\left(\frac{1}{r^{\frac{1}{p}}}\right).$$

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This implies that for each $1 \leq p < \infty$ there are $d_p, D_p > 0$ and $r_0 > 0$ such that for each $|\theta| = 1$ and each $r > r_0$

$$d_p \frac{e^r}{r^{\frac{1}{2p}}} \leq M_p(e^{\theta z}, r) \leq D_p \frac{e^r}{r^{\frac{1}{2p}}} \tag{1}$$

Theorem

Let v be a weight function such that $\lim_{r\to\infty} v(r)\frac{e^r}{r^{\frac{1}{2p}}} = 0$ for some $1 \leq p \leq \infty$. If the differentiation operator $D : B_{p,0} \to B_{p,0}$ is continuous, then D is frequently hypercyclic.

Idea of the proof

We must to prove for any countable set $A \subset \mathbb{T}$,
$E_{\mathbb{T}\setminus A} = \mathrm{span}(\{e^{\theta z} : |\theta| = 1, \theta \in \mathbb{T} \setminus A\})$ is contained and dense in $B_{p,0}$.

Idea of the proof

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To prove the density, we define the following vector valued functions on the closed unit disc $\overline{\mathbb{D}}$:

$$H : \overline{\mathbb{D}} \to B_{p,0},\ H(\zeta)(z) := e^{\zeta z},\ \zeta \in \overline{\mathbb{D}}.$$

The function H is well defined, H is holomorphic on $\mathbb{D}$ and $H : \overline{\mathbb{D}} \to B_{p,0}$ is continuous.

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Since the function $u \circ H$ is holomorphic in $\mathbb{D}$, continuous at the boundary and vanishes at the points $\zeta \in \mathbb{T} \setminus A$, it is zero in $\mathbb{D}$. In particular $(u \circ H)^{(n)}(0) = u(H^{(n)}(0)) = u(z^n) = 0$, hence u vanishes on the polynomials. As the polynomials are dense in $B_{p,0}$, we conclude $u = 0$.

Corollary

Let $\varphi(r)$ be a positive function with $\lim_{r\to\infty}\varphi(r)=\infty$. For each $1\leq p\leq\infty$ there is an entire function f such that

$$M_p(f,r)\leq\varphi(r)\frac{e^r}{r^{\frac{1}{2p}}}$$

that is frequently hypercyclic for the differentiation operator D on $H(\mathbb{C})$.

Proof

It is possible to find a positive increasing continuous function $\psi(r) \leq \varphi(r)$ with $lim_{r\to\infty}\psi(r) = \infty$ and $\sup\limits_{r>0} \dfrac{\psi(r+1)}{\psi(r)} < \infty$.

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Define $v(r) = \dfrac{r^{\frac{1}{2p}}}{\psi(r)e^r}$ for $r \geq r_0$, with r_0 large enough to ensure that $v(r)$ is non increasing on $[r_0, \infty[$, and $v(r) = v(r_0)$ on $[0, r_0]$. One can take $r_0 = \frac{1}{2p}$.

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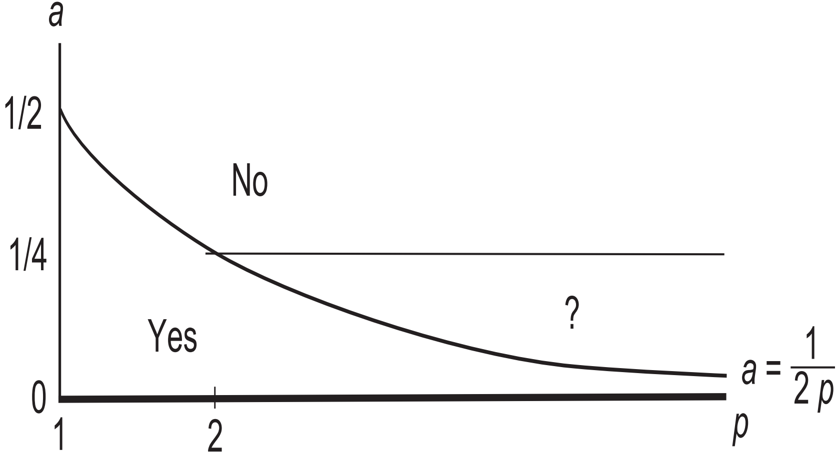

Theorem (Drasin-Saksman, in a couple weeks in ArXiv)

For any $C > 0$ there is an entire function D-frequently hypercyclic f with

$$M_p(f,r) \leq C\frac{e^r}{r^{a(p)}}$$

where $a(p) = \frac{1}{4}$ for $p \in [2,\infty]$ and $a(p) = \frac{1}{2p}$ for $p \in (1,2]$

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The construction is direct with no functional analysis, but uses remarkable polynomials of Rudin-Shapiro and de la Vallee Poussin.(Conference in honour of P. Gauthier and K. Gowrisankaran, june 20-23, 2011)