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CESARO SUMMABILITY

OF

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JOINT WORK WITH:

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DEFINITIONS

Let $S_n(f,\xi)(z)=\sum_{k=0}^{n} c_k(z-\xi)^k, n=0,1,2,...,$

be the sequence of partial sums of the Taylor development of

$f(z)=\sum_{k=0}^{\infty} c_k(z-\xi)^k$ in the open disk $D(\xi,R)$, $0 < R < \infty$.

If $K \subset \mathbb{C}$ with $K \cap D(\xi,R)=\emptyset$, then we say that the function f (or the Taylor series $\sum_{k=0}^{\infty} c_k(z-\xi)^k$) is universal with respect to K (or K–universal), if for every function

$h: K \to \mathbb{C}$ continuous on K and holomorphic in the interior

of K (if $K^o \neq \emptyset$), there exists a sequence of natural numbers

$\{k_n\}, n \in \mathbb{N}$, such that the subsequence $S_{k_n}(f,\xi)(z)$ converges

to h(z) uniformly on K.

Let f be a K–universal function, $z_0 \in$K and $a > -1$. We

say that f is $(C,a)-$summable at z_0 to a finite sum s, if we

have that:

$$\sigma_n^a(f,\xi)(z_0) \xrightarrow[n\to\infty]{} s,$$

for the sequence of a-Cesaro means

$$\sigma_n^a(f,\xi)(z_0)=\sum_{k=0}^{n}\frac{A_{n-k}^a}{A_n^a}c_k(z_0-\xi)^k,$$

where $A_m^a=\dfrac{(a+1)(a+2)\cdots(a+m)}{m!}$.

EXAMPLES

1. Let $\Omega\neq\mathbb{C}$ be a <u>simply connected domain</u> in $\mathbb{C}$ and $\xi\in\Omega$. We denote by $H(\Omega)$ the set of holomorphic functions on Ω, endowed with the topology of uniform convergence on compact sets. We also denote by K=c.c.c. any subset K of $\mathbb{C}$, which is Compact with Connected Complement. Then:

(a) The following class of universal functions

$$U_1(\Omega,\xi)=\left\{f\in H(\Omega):f(z)=K-universal,\forall K=c.c.c.,K\cap\bar{\Omega}=\varnothing\right\}$$

is a $G_\delta-$dense subset of $H(\Omega)$. These classes studied, initially in the case where $\Omega=D(0,1)$, by Luh and independently by Chui and Parnes – in the early of 70's.

Universal functions in $U_1(D,0)$ have as natural boundary the unit circle $T=\partial D(0,1)=\left\{z\in\mathbb{C}:|z|=1\right\}$, but they may be smooth on T.

There are functions in this class, which have the highest order of regularity and summability at the boundary of D(0,1); their Taylor developments are (C,a)-summable for every $a>-1$, at any point of T.

(b) In 1996, V. Nestoridis strengthened the results of Luh and Chui – Parnes by allowing to the compact set K to meet T and the universal approximation was obtained on the boundary of D(0,1) as well. Then, the following new class of universal functions, defined by

$$U(\Omega,\xi)=\left\{ f\in H(\Omega):\ f(z)=K-universal,\forall K=c.c.c.,K\cap\Omega=\varnothing \right\},$$

is again a $G_\delta-$dense subset of $H(\Omega)$. It is a proper subset of the previous class and the universal functions in $U(\Omega,\xi)$ have several wild properties. For instance, when $\Omega=D(0,1)$, their Taylor coefficients can not have polynomial growth and so their Taylor development can not be $(C,a)-$summable, for every $a>-1$, at any point of T. This argument gives the $(C,a)-$non-summability simultaneously for all points of T and it is not possible to have the result for some such points. There is a different proof of this fact, based on an extension of Rogosinski' s formula, which enables one to distinguish some points on T and so we shall follow this method here.

2. There are unbounded non-simply connected domains Ω in $\mathbb{C}$, where universal Taylor series exist. On the other hand, if Ω is a bounded annulus the class of universal functions is empty. This leads us to look for weaker approximation results on arbitrary planar domains.
From their main results we need the following:
Let $\Omega\neq\mathbb{C}$ be an arbitrary planar domain, $\xi\in\Omega$,

$$R=dist\left(\xi,\partial\Omega\right)>0,$$

$$J(\Omega,\xi)=C(\xi,R)\cap\partial\Omega,$$

and $C(\xi,R)=\left\{z\in\mathbb{C}:\left|z-\xi\right|=R\right\}$.
Then, we define the following class of universal functions:

$$B(\Omega,\xi)=\left\{ f\in H(\Omega):\ f(z)=K-universal,\forall K=c.c.c.,K\subset J(\Omega,\xi) \right\}$$

This class is again a $G_\delta-$dense subset of $H(\Omega)$.

If $J(\Omega,\xi)$ contains an arc of $C(\xi,R)$ with strictly positive opening, then the Taylor coefficients of any $f\in B(\Omega,\xi)$ can not have polynomial growth and so the

Taylor development of f, with center ξ, can not be $(C,a)-$summable for any $z \in J(\Omega,\xi)$ and every $a > -1$.

CESARO SUMMABILITY IN THE GENERAL CASE

If $f(z) = \sum_{k=0}^{\infty} c_k (z-\xi)^k$ ($z \in D(\xi,R)$, $0 < R < \infty$) is a universal function with respect some $K \subset \mathbb{C}$, $K \cap D(\xi,R) = \varnothing$ and $z_0 \in K$, then the question is

<u>under what conditions on K and z_0 the above Taylor series is not $(C,a)-$summable at z_0 to a finite sum ($a > -1$)</u>?

It is well known that for z_0 with $|z_0 - \xi| > R$ the above series is not $(C,a)-$summable at z_0, for every $a > -1$. So, from now on, we assume that $z_0 \in K \cap C(\xi,R)$. Moreover, without loss of generality we may assume that:

$$D(\xi,R) = D(0,1), \quad K \subset T \quad \text{and} \quad z_0 = 1 \in K \cap T.$$

We recall now the classical formula of Rogosinski:

<u>Rogosinski's formula</u>: $z_n \in \mathbb{C}$, $n \in \mathbb{N}$: $1 - z_n = O(\frac{1}{n})$ and $\sum_{k=0}^{\infty} c_k$ is $(C,1)-$summable to $s \in \mathbb{C}$. If $S_n(z) = \sum_{k=0}^{n} c_k z^k$, then we have:

$$S_n(z_n) - s - \left(S_n(1) - s\right) z_n^n \xrightarrow[n\to\infty]{} 0.$$

From this we obtain easily the following:

<u>Proposition 1</u>: Let $K \subset T$, $1 \in K$ and $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be a K-universal function. If there exists a sequence $\{z_n\} \subset K$ such that:

(i) $1-z_n = O(\frac{1}{n})$ **and**

(ii) $\{z_n^n\}$ **has not 1 as a limit point,**

then the series $\sum_{k=0}^{\infty} c_k$ **is not (C,1) – summable.**

<u>Proof</u>:

Suppose that the series $\sum_{k=0}^{\infty} c_k$ **is (C,1) – summable to** $s\in\mathbb{C}$ **. Put h(z)=s+1, z∈K. Since** f **is K-universal there exists a sequence** $\{k_n\}\subset\mathbb{N}$ **, such that** $S_{k_n}(z)\xrightarrow[n\to\infty]{}$ **h(z)=s+1 uniformly on K. Then, Rogosinski's formula implies that**

$$z_{k_n}^{k_n}\xrightarrow[n\to\infty]{} 1, \text{ a contradiction by (ii).}$$

Since we used Rogosinski's formula, condition (i) in Prop. 1 is necessary.

<u>The question is if condition (ii) is necessary, or we can replace it by a weaker one</u>.

Of course, there are stronger conditions which imply conditions (i) and especially (ii) and we give two such examples.

(C.1) If $\hat{K}=\{t\in[-\pi,\pi]: z=e^{it}\in K\}$**, then 0 is a right (resp. left) density point of** $\hat{K}$ **.**

(C,1) implies also the following condition (C.2), which in turn implies (i) and (ii).

(C.2) $\exists\ z_n = e^{it_n}\in$ **K such that (a)** $n|1-z_n|\xrightarrow[n\to\infty]{}$ **c > 0 and (b)** $\exists\ n_0\in\mathbb{N}$ **:** $n|t_n|<c_1<2\pi$ **,** $\forall n\ge n_0$ **.**

<u>Remark:</u> As we shall see, (a) of (C.2) <u>guarantees</u> the (C,1) – non – summability of the series $\sum_{k=0}^{\infty} c_k$ **. Thus, for example, if** $z_n = exp\left(\frac{2\pi i}{n}\right)\in$**K, then** $n|1-z_n|\xrightarrow[n\to\infty]{} 2\pi$ **> 0 and so** $\sum_{k=0}^{\infty} c_k$ **is**

not (C,1) – summable. Moreover, $z_n^n = exp(2\pi i) = 1$, which implies that the answer to the previous question is negative and condition (ii) is <u>not</u> necessary.

Now, if we examine more carefully the above conditions and especially (a) of (C.2), we see that the following condition (C*) is a (weaker) conclusion of (a) of (C.2) and it is sufficient for our purposes.

Let $\hat{K} = \{t \in [-\pi, \pi] : z = e^{it} \in K\}$. We assume that:

$$\forall\, N \in \mathbb{N},\ N \geq 1 \text{ and } \forall\, S \subset \mathbb{N},\ S \text{ infinite,}$$

(C*) $$\exists\, a_N = a_N(S),\ b_N = b_N(S) \text{ with } 0 < a_N < b_N < \frac{1}{N},$$

$$\exists\, \{k_n\} \subset S \text{ and } \exists\, \{t_n\} \subset \hat{K} \text{ such that}$$

$$k_n |t_n| \in (a_N, b_N), \text{ for every n=1,2,3,….}$$

Then, we have the following:

<u>Theorem:</u> Let $K \subset T$, $1 \in K$ and $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be a K-universal function. If K satisfies condition (C*), then the series $\sum_{k=0}^{\infty} c_k$ is not (C,a) – summable for every $a > -1$.

It is sufficient to prove this Theorem for $a =$ m, an arbitrary <u>natural</u> number, since, as it is well known, if a series is (C,m)-summable, then it is (C,a)–summable, for every $a \geq$ m.

(So, the (C,m)-non-summability, for every m$\in \mathbb{N}$, implies the (C,a)-non-summability for every $a > -1$).

We shall not give the proof here, which is much more complicated than that of Prop. 1, because of the complexity

of condition (C*) and the use of the following extension of Rogosinski's formula:

Let $\sum_{k=0}^{\infty} c_k$ be (C,m)–summable for some natural number m, to a finite sum s and let

$$S \subset \mathbb{N},\ S \text{ infinite and}$$
$$\forall n \in S,\ \forall j=1,2,\dots,m,$$
$$\exists z_{j,n} \in \mathbb{C} \text{ such that}$$
$$\lim_{n\to\infty} n\left(1-z_{j,n}\right)=\varsigma_j, \text{ with } 0<|\varsigma_1|<|\varsigma_2|<\cdots<|\varsigma_m|.$$

Then,

$$\forall n \in S,\ \forall j=1,2,\dots,m,$$
$$\exists \lambda_{j,n} \in \mathbb{C} \text{ such that}$$
$$\sum_{j=1}^{m}\left[\frac{1}{z_{j,n}^{n}}\left(S_n(z_{j,n})-s\right)-\left(S_n(1)-s\right)\right]\lambda_{j,n} \xrightarrow[n\in S,\, n\to\infty]{} 0 \quad \text{and}$$
$$\lim_{n\in S,\, n\to\infty} \lambda_{j,n}=\frac{1}{\varsigma_j \prod_{k\neq j}\left(\varsigma_j-\varsigma_k\right)}.$$

On the other hand, it seems that the situation is quit different in the case where the set K is only a <u>finite</u> subset of $\mathbb{C}$, say K$=\{z_1,z_2,\cdots,z_n\}$, with $|z_j|\geq 1$, j=1,2,…,n.

Then, there are K–universal functions $f(z)=\sum_{k=0}^{\infty} c_k z^k$ in D(0,1), which are (C,a)–summable at every point $z_j \in K \cap T$ and every $a \geq 1$.

For example, let K={1} and let $\{q_n\}$, n=0,1,2,…, be an enumeration of the rational points of $\mathbb{C}$, i.e. $q_n \in \mathbb{Q}+\mathbb{Q}i$. We choose a sequence $\{\lambda_n\}$ with $\lambda_0=0$, $\lambda_{n+1}>\lambda_n+m$, where m is some suitable natural number and moreover

$$\frac{|q_0+q_1+\cdots+q_n|}{\lambda_n+1}<\frac{1}{n+1}, \text{ for every natural number n.}$$

Then we define

$$f(z)=\sum_{n=0}^{\infty}\left(q_n-q_n z\right)z^{\lambda_n},$$

which, for m$\geq$2, has the following properties:

(i) $S_{\lambda_n}(1)=q_n$, **n=0,1,2,….**

(ii) $\dfrac{\left|S_0+S_1+\cdots+S_k\right|}{k+1}=\dfrac{\left|q_0+q_1+\cdots+q_n\right|}{k+1}<\dfrac{1}{n+1},$

for every $k\in\mathbb{N}$ **with** $\lambda_n\leq k<\lambda_{n+1}$ **and every n=0,1,2,….**

Since $\{q_n\}$ **is dense in** $\mathbb{C}\cup\{\infty\}$**, it follows from (i) that** $f(z)=\sum_{k=0}^{\infty}c_k z^k$**, is a K–universal function with radius of convergence R$\leq$1. By (ii) we have that the Taylor series of** f **is (C,1)–summable, at** z_0**=1, to the finite sum 0 and so R=1. Moreover, for every** $a\geq$**1 the series** $\sum_{k=0}^{\infty}c_k$ **is (C,**a**)–summable.**

Further, if m$\geq$4, we can consider the function

$$f(z)=\sum_{n=0}^{\infty}\left(q_n-q_n z-q_n z^2+q_n z^3\right)z^{\lambda_n},$$

which is K–universal and has a K–universal and (C,1)–summable, at z_0**=1, derivative and so on.**

In the general case where K$=\{z_1,z_2,\cdots,z_n\}$, with $\left|z_j\right|\geq$**1, j=1,2,…,n, the construction of K–universal functions** $f(z)=\sum_{k=0}^{\infty}c_k z^k$ **in D(0,1), which are (C,**a**)–summable at every point** $z_j\in K\cap T$ **and every** $a\geq$**1, is similar to the above example. In fact, there are two ways to construct such examples. One, by using the following result of Dirichlet:**

If $\{z_1,z_2,...,z_m\}\subset$**T, then (1,1,…,1) is a limit point of** $\left\{\left(z_1^n,z_2^n,...,z_m^n\right),n=1,2,3,...\right\}$**,**

and second without this, but with some more complicated technique.