Topological and algebraic structure of the set of strongly annular functions

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Joint work with Antonio Bonilla

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- Hence $f \in \mathcal{SA} \iff \exists$ a sequence of circles $C_n = \{|z| = r_n\}$ in $\mathbb{D}$ with $r_n \uparrow 1$ such that $\lim_{n\to\infty} \min\{|f(z)| : z \in C_n\} = +\infty$.

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  Then $\prod_{k=1}^{\infty}(1 - \frac{3z^{n_k}}{a_k^{n_k}}) \in \mathcal{SA}$.

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  Then $\prod_{k=1}^{\infty}(1 - \frac{3z^{n_k}}{a_k^{n_k}}) \in \mathcal{SA}$.
- An example with power series: Take
  $0 < a_n < b_n < a_{n+1} < \cdots \to 1$ and choose $(n_k) \uparrow \infty$ such that $(r_k/s_k)^{n_k} \geq k(1 + \sum_{j=1}^{k-1}(r_k/s_j)^{n_j})$ and $(r_{k-1}/s_k)^{n_k} \leq 1/2^k$ $(k \geq 1)$. Then $\sum_{k=1}^{\infty}(z/s_k)^{n_k} \in \mathcal{SA}$.

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- There are also explicit constructions of series $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{SA}$ with $a_n \to 0$.
  [Bonar, Carroll and Piranian, 1977].

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- To sum up: $\mathcal{SA}$ is topologically large.

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Theorem 1: Assume that Y is a Baire topological vector space with $Y \subset H(\mathbb{D})$ such that Y is endowed with a topology τ which is finer that $\tau_c|_Y$. If $\mathcal{SA} \cap Y \neq \emptyset$ and there is a dense subset $\mathcal{D}$ of Y such that each function $f \in \mathcal{D}$ is bounded on $\mathbb{D}$, then $\mathcal{SA} \cap Y$ is residual in Y.

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- Concrete examples?
- Hardy spaces $H^p(\mathbb{D}) := \{f \in H(\mathbb{D}) : \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < +\infty\}$ $(p > 0)$ and $H^\infty(\mathbb{D})$ are discarded, due to Fatou's theorem.
  In fact, $H^p(\mathbb{D}) \cap \mathcal{SA} = \emptyset$.

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- A positive example: In 2007, Redett constructed a SA-function in each generalized Bergman space, defined for $0 < p < +\infty$, $\alpha > -1$ as $A^p_\alpha(\mathbb{D}) := \{f \in H(\mathbb{D}) : \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p (1-r)^\alpha r \, d\theta dr < +\infty\}$.

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- For the construction, Redett used the following result [Buckley, Koskela and Vukotic, 1999]:
  Let $(p_n) \subset \mathbb{N}$ with $p_{n+1} > 2p_n\ (n \geq 1)$, and $f(z) = \sum_{n=0}^\infty a_n z^{p_n} \in H(\mathbb{D})$. Then $f \in A^p_\alpha(\mathbb{D})$ if and only if $\sum_{n=1}^\infty |a_n|^p p_n^{-\alpha-1} < +\infty$.

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- Note that $A^p_\alpha(\mathbb{D})$ is a Fréchet space [even a Banach space if $p \geq 1$, and a Hilbert space if $p = 2$].

Corollary: $\mathcal{SA} \cap A^p_\alpha(\mathbb{D})$ is residual in $A^p_\alpha(\mathbb{D})$.

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Theorem 2: For σ and φ as above, the set $\mathcal{SA}(\varphi, \sigma) :=$

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- In the first decade of the present millenium, Aron, Bayart, Gurariy, Seoane, Quarta and LBG coined the following notions.

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- spaceable whenever $A \cup \{0\}$ contains a closed infinite dimensional vector subspace of X, and
- algebrable if X is a function space and $A \cup \{0\}$ contains some infinitely generated algebra.

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Lemma A [LBG, 2010]: Assume that X is a metrizable separable TVS. Suppose that Γ is a family of linear subspaces of X such that $\bigcap_{S\in\Gamma} S$ is dense in X and $\bigcap_{S\in\Gamma}(X \setminus S)$ is μ-lineable, where μ is an infinite cardinal number. Then $\bigcap_{S\in\Gamma}(X \setminus S) \cup \{0\}$ contains a dense μ-dimensional VS.

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Sketch of proof: Use Buckley–Koskela–Vukotic's result to produce a power series $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{SA} \cap A^p_\alpha(\mathbb{D})$ [coefficients a_n should be bigger enough than $a_0, \dots, a_{n-1}$ but not too much!] such that after an infinite partitioning, the resulting f_n's are still in $\mathcal{SA}$. Then

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Apply Lemma A with $\mu =$ card$(\mathbb{N})$, $X = A^p_\alpha(\mathbb{D})$ and
$\Gamma = \{\langle f\rangle + \{$polynomials$\} : \, f \in A^p_\alpha(\mathbb{D}) \setminus \mathcal{SA}\}$. $\square$

- In fact, each $\mathcal{SA}(\varphi, \sigma)$ is maximal dense-lineable and algebrable ... but we do not know whether or not these properties are true for $\mathcal{SA} \cap A^p_\alpha(\mathbb{D})$ [$H(\mathbb{D})$-proofs do not adapt].

Problem: Are these sets spaceable?

[Recall: $A \subset X$ TVS is spaceable if $\exists$ closed VS $M \subset X$ with dim$(M) = +\infty$ and $M \subset A \cup \{0\}$]

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Theorem 6: $\mathcal{SA}_e$ is residual and algebrable.
The same for each family $\mathcal{SA}_e(\varphi, \sigma)$.

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Sketch of proof: The trick is to demonstrate the property for the (smaller) class $S(\gamma) :=$ $\{f \in H(\mathbb{C}) : \limsup_{r\to\infty} \frac{m(f,r)}{e^{r^\gamma}} = +\infty\}$, where $\gamma \in (0, 1/2)$.

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