A quantitative estimate for bounded point evaluations in $P^2(\mu)$-spaces

joint with S. Richter and C. Sundberg

Given a finite positive Borel measure μ on $\mathbb{C}$ $P^2(\mu)$ is the closure of analytic polynomials in $L^2(\mu)$.

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Examples

- If μ is supported on $[0,1]$ then
$$P^2(\mu) = L^2(\mu)$$
by the Weierstrass's approximation theorem.

- If μ is the arclength measure on a circle, then
$$P^2(\mu) \neq L^2(\mu).$$
In fact, $P^2(\mu)$ can be identified with a space of analytic functions. That space is called the Hardy space on the disc bounded by this circle.

- If $\mu =$ area measure (dA) on a disc $P^2(\mu)$ is again a space of analytic functions called the Bergman space on that disc.

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Examples

- If $\mu =$ area measure (dA) on a disc + arclength measure on the boundary of a disjoint disc + any measure ν supported on a segment that does not intersect the 2 discs then: $P^2(\mu)$ = the Bergman space on the first disc $\oplus$ the Hardy space on the second disc $\oplus$ $L^2(\nu)$.

In 1991 J. Thomson proved the following remarkable result.

Theorem

Any finite positive Borel measure μ on $\mathbb{C}$ can be decomposed

$$\mu = \sum_{j=0}^{\infty} \mu_j \,, \quad \mu_i \perp \mu_j \, i \neq j \,,$$

such that

$$P^2(\mu) = L^2(\mu_0) \oplus \left(\bigoplus_{j=1}^{\infty} P^2(\mu_j) \right) \,,$$

where $P^2(\mu_j)$ are Hilbert spaces of analytic functions on pairwise disjoint simply connected regions Ω_j.

In general, the Hilbert spaces of analytic functions $P^2(\mu_j)$ are quite difficult to understand.
Assume for simplicity that

$$\Omega_j = \mathbb{D} = \text{the unit disc}$$

and let $\mathbb{T}$ be the unit circle.

Theorem

(i) If $\mu_j(\mathbb{T}) > 0$ then each function in $P^2(\mu_j)$ has nontangential limits a.e. on the support of μ and the restriction of the shift operator $f \to zf$ to any invariant subspace has Fredholm index -1.
(ii) If $\mu_j(\mathbb{T}) = 0$ then there exist functions in $P^2(\mu_j)$ that have no nontangential limits a.e., and the Fredholm index of the restriction of the shift operator to an invariant subspace can be any negative integer.

Bounded point evaluations

A point $\lambda \in \mathbb{C}$ is a bounded point evaluation for $P^2(\mu)$ if *evaluation at* λ is a bounded linear functional on $P^2(\mu)$.

In other words, there exists $c_\lambda > 0$ such that

$$(*) \qquad |p(\lambda)| \le c_\lambda \left(\int |p|^2 d\mu \right)^{1/2} = \|p\|_{L^2(\mu)},$$

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The following weaker version of Thomson's theorem actually is the heart of the matter:

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This extends the celebrated Scott Brown theorem (1978) that subnormal operators have nontrivial invariant subspaces.

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Note that the above theorem
$P^2(\mu) \neq L^2(\mu) \iff$ *there are bounded point evaluations*

- has a fairly weak hypothesis:
  There exists a nonzero $G \in L^2(\mu)$ with

$$\int pGd\mu = 0\,,$$

  for all polynomials p.

- Its conclusion seems completely unrelated:
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- **Where are these points located?**

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A naive approach

If $G \in L^2(\mu) \setminus \{0\}$ satisfies

$$\int pG d\mu = 0$$

for all polynomials p, then

$$\int \frac{p - p(\lambda)}{z - \lambda} G d\mu = 0 \,,$$

whenever p is a polynomial and $\lambda \in \mathbb{C}$.

Rewrite this

$$p(\lambda) \int \frac{1}{z - \lambda} G d\mu = \int \frac{1}{z - \lambda} pG d\mu. \tag{1}$$

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Rewrite this

$$(1) \qquad p(\lambda) \int \frac{1}{z - \lambda} G d\mu = \int \frac{1}{z - \lambda} pG d\mu.$$

For a compactly supported finite complex measure ν on $\mathbb{C}$ let $C(\nu)$ be its *Cauchy transform*

$$C(\nu)(\lambda) = \int \frac{d\nu}{z-\lambda}\,.$$

The corresponding potential is defined by

$$U_{|\nu|}(\lambda) = \int \frac{d|\nu|}{|z-\lambda|}\,.$$

The equality (1) can be rewritten as:

$$(2) \qquad p(\lambda)C(G\mu)(\lambda) = C(pG\mu)(\lambda)\,,$$

where, by abuse of notation $dG\mu = Gd\mu,\ dpG\mu = pGd\mu$.

If $C(G\mu)(\lambda) \neq 0$ and

$$U_{|pG\mu|}(\lambda) \lesssim \|p\|_{L^2(\mu)}$$

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If $C(G\mu)(\lambda) \neq 0$ and

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then λ is a bounded point evaluation.

For example, such a G can be found if there exists $\varepsilon > 0$ such that

$$P^2(\mu|\{|z-\lambda| > \varepsilon\}) \neq L^2(\mu|\{|z-\lambda| > \varepsilon\}).$$

The answer to the previous questions about bounded point evaluations does involve the above objects, but a more sophisticated approach. The following result can be deduced from the proof of Theorem 2, or alternatively from J. Brennan's work.
The formulation below together with another proof is due to J. Thomson.

Theorem

Let G be as above. If

$$C(G\mu)(\lambda) \neq 0\,, \qquad U_{|G\mu|}(\lambda) < \infty$$

then λ is a bounded point evaluation for $P^2(\mu)$.

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then λ is a bounded point evaluation for $P^2(\mu)$.

Brennan's trick

This uses averages of the Cauchy transforms involved in (2). Recall that $p(\lambda)C(G\mu)(\lambda) = C(pG\mu)(\lambda)$, G annihilates analytic polynomials.

Lemma

Let G be as above. and let $d\nu = Gd\mu$. If $r, K_0 > 0$ such that

$$|p(\lambda)C(\nu)(\lambda)| \leq \frac{K_0}{r^2}\int_{\{|z-\lambda|<r\}} |C(p\nu)|dA = \frac{K_0}{r^2}\int_{\{|z-\lambda|<r\}} |pC(\nu)|dA,$$

then

$$|p(\lambda)C(\nu)(\lambda)| \leq \frac{2\pi K_0}{r}\|G\|_{L^2(\mu)}\|p\|_{L^2(\mu)}.$$

Brennan's trick

Thus if $C(\nu)(\lambda) \neq 0$, the inequality

$$|p(\lambda)C(\nu)(\lambda)| \lessapprox \int_{\{|z-\lambda|<r\}} |pC(\nu)|dA\,,$$

implies that λ is a bounded point evaluation! Here $d\nu = Gd\mu$ and G annihilates the polynomials.

Main result

It turns out that such an estimate holds true for any compactly supported finite Borel measure ν!
In particular, it easily implies Theorem 4.

Theorem

There exists an absolute constant $K > 0$ such that for every compactly supported finite Borel complex measure ν on $\mathbb{C}$ and for every $\lambda \in \mathbb{C}$ with $U_{|\nu|}(\lambda) = \int \frac{1}{|z-\lambda|} d|\nu|(z) < \infty$ there exists $r_0 > 0$ such that for all polynomials p and for all $0 < r \leqslant r_0$ we have

$$|p(\lambda)C(\nu)(\lambda)| \leqslant \frac{K}{r^2} \int_{\{|z-\lambda|<r\}} |pC(\nu)| dA .$$

An outline of the proof.

1) Analytic capacity.
For a compact $K \subseteq \mathbb{C}$ we define the *analytic capacity* of K by

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C}_\infty \setminus K), \|f\|_\infty \leqslant 1\},$$

where

$$f'(\infty) = \lim_{z \to \infty} z[f(z) - f(\infty)].$$

An outline of the proof.

A related capacity, γ_+, is defined as follows:

Consider the set $\Sigma_\infty(K)$ consisting of nonnegative finite Borel measures σ supported on K such that

$$C(\sigma) \in L^\infty(\mathbb{C}), \quad |C(\sigma)(z)| \leqslant 1 \text{ for A-a.e. } z \in \mathbb{C},$$

and set

$$\gamma_+(K) = \sup\{\sigma(K) : \sigma \in \Sigma_\infty(K)\}.$$

For a measure $\sigma \in \Sigma_\infty(K)$ we have that $C(\sigma)$ is analytic in $\mathbb{C}_\infty \setminus K$ and $(C(\sigma))'(\infty) = -\sigma(K)$, hence

$$\gamma_+(K) \leqslant \gamma(K)$$

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Tolsa's theorem.

In 2001, Tolsa proved the astounding result that γ_+ and γ are actually equivalent:

Theorem

There is an absolute constant A_T such that

$$\gamma(K) \leqslant A_T \gamma_+(K)$$

for all compact sets $K \subseteq \mathbb{C}$.

An outline of the proof.

The following weak-type inequality for analytic capacity is essentially a direct consequence of Tolsa's theorem:

Lemma

Suppose $\sigma = \omega dA$, where ω is a compactly supported bounded function. We then have

$$\gamma(\{\Re C(\sigma) \geq a\}) \leq \frac{A_T}{a}\|\sigma\| \qquad \textit{for all } a > 0\,.$$

An outline of the proof.

2) A lemma about bounded point evaluations and analytic capacity

Thomson's famous coloring scheme can be used to prove the following interesting estimate:

Lemma

There are absolute constants $\varepsilon_1 > 0$ and $K_1 < \infty$ with the following property. Let $\lambda \in \mathbb{C}$, $r > 0$ and let

$$E \subset \{|z - \lambda| < r\}$$

be compact with $\gamma(E) < r\varepsilon_1$. Then

$$|p(\lambda)| \leq \frac{K_1}{r^2} \int_{\{|z-\lambda|<r\}\setminus E} |p| \frac{dA}{\pi},$$

for all polynomials p.

An outline of the proof.

Recall that in order to prove Theorem 6, we need to show that

$$|p(\lambda)C(\nu)(\lambda)| \leqslant \frac{K}{r^2}\int_{\{|z-\lambda|<r\}} |pC(\nu)|dA\,,$$

for an arbitrary finite measure ν with compact support and all polynomials p.

An outline of the proof.

Write $\nu = \nu_1 + \nu_2$, where

$$\nu_1 = \nu|\{|z-\lambda| \geq 2r\}, \quad \nu_2 = \nu|\{|z-\lambda| < 2r\}.$$

Assume that $C(\nu)(\lambda) = a > 0$ and that $U_{|\nu|}(\lambda) < \infty$.

An outline of the proof.

The assumption $U_{|\nu|}(\lambda) < \infty$ implies that

$$\frac{\|\nu_2\|}{r} \le 2 \int_{\{|z-\lambda|<2r\}} \frac{d|\nu|}{|z-\lambda|} = o(1)\,,$$

when $r \to 0$.

Thus by Lemma 8, if r is small, $\nu = \omega dA$ with ω compactly supported and bounded, then

$$\gamma(\{\Re C(-\nu_2) \ge a/3\}) < r\varepsilon_1\,,$$

where ε_1 is the constant in Lemma 9.

An outline of the proof.

A standard Bergman-space estimate gives

$$|C(\nu_1)(z) - a| < \frac{1}{2}a \Rightarrow \Re C(\nu_1)(z) > \frac{a}{2} \quad |z - \lambda| < r\,.$$

An outline of the proof.

If $E = \{\Re C(-\nu_2) \geq a/3\}$ then by Lemma 9

$$\begin{aligned}\int_{\{|z-\lambda|<r\}} |pC(\nu)|dA &\geq \int_{\{|z-\lambda|<r\}\setminus E} |p|\Re C(\nu)dA \\ &\geq \frac{a}{6}\int_{\{|z-\lambda|<r\}\setminus E} |p|dA \\ &\geq \frac{r^2}{K_1}|p(\lambda)|,\end{aligned}$$

and the result follows.