# Remarks on weighted mixed norm spaces

**OSCAR BLASCO**

[1]Departamento Análisis Matemático
Universidad Valencia

*Charm2011-Málaga 13 July 2011*

## The operators

Given $\alpha > -1$ one defines the **Bergman-type projection** $P_\alpha$ by the formula

$$P_\alpha(f)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha f(w)}{(1-\bar{w}z)^{2+\alpha}}\, dA(w) \tag{1}$$

for any $f \in L^1((1-|w|^2)^\alpha dA(w))$.

## The operators

Given $\alpha > -1$ one defines the **Bergman-type projection** $P_\alpha$ by the formula

$$P_\alpha(f)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha f(w)}{(1-\bar{w}z)^{2+\alpha}}\,dA(w) \tag{1}$$

for any $f \in L^1((1-|w|^2)^\alpha dA(w))$.

## The operators

Given $\alpha > -1$ one defines the **Bergman-type projection** $P_\alpha$ by the formula

$$P_\alpha(f)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha f(w)}{(1-\bar{w}z)^{2+\alpha}}\,dA(w) \tag{1}$$

for any $f \in L^1((1-|w|^2)^\alpha dA(w))$.
Given a function $\varphi \in L^1(\mathbb{D})$, its **Berezin transform** is defined by

$$\widetilde{\varphi}(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\varphi(w)}{|1-\bar{w}z|^4}\,dA(w) \tag{2}$$

## The operators

Given $\alpha > -1$ one defines the **Bergman-type projection** $P_\alpha$ by the formula

$$P_\alpha(f)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha f(w)}{(1-\bar{w}z)^{2+\alpha}}\, dA(w) \tag{1}$$

for any $f \in L^1((1-|w|^2)^\alpha dA(w))$.
Given a function $\varphi \in L^1(\mathbb{D})$, its **Berezin transform** is defined by

$$\widetilde{\varphi}(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\varphi(w)}{|1-\bar{w}z|^4}\, dA(w) \tag{2}$$

For a fixed $0 < \alpha < \pi$, the **averaging operator** of $\varphi$ is given by

$$A_\alpha(\varphi)(z) = \frac{1}{|A(z,\alpha)|}\int_{A(z,\alpha)} \varphi(\omega) dA(\omega), \tag{3}$$

where $A(z,\alpha) = \{w : |z| < |w| < 1, |Arg(w) - Arg(z)| < \alpha(1-|z|)\}$ is the Carleson box at the point $z$.

## Mixed norm spaces

Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$ the mixed norm spaces $L^{p,q,\alpha}$ consist of all measurable complex functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{L^{p,q,\alpha}} = \left( \int_0^1 (1-r^2)^{q\alpha-1} M_p^q(f,r) dr \right)^{1/q} < \infty, \tag{4}$$

with

$$M_p(f,r) = \left( \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

## Mixed norm spaces

Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$ the mixed norm spaces $L^{p,q,\alpha}$ consist of all measurable complex functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{L^{p,q,\alpha}} = \left( \int_0^1 (1-r^2)^{q\alpha-1} M_p^q(f,r) dr \right)^{1/q} < \infty, \tag{4}$$

with

$$M_p(f,r) = \left( \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

In particular,
$L(p,p,1/p) = L^p(\mathbb{D}, dA), L(p,p,\beta/p) = L^p(\mathbb{D}, (1-|z|)^{\beta-1} dA).$

## Mixed norm spaces

Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$ the mixed norm spaces $L^{p,q,\alpha}$ consist of all measurable complex functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{L^{p,q,\alpha}} = \left( \int_0^1 (1 - r^2)^{q\alpha - 1} M_p^q(f, r) dr \right)^{1/q} < \infty, \tag{4}$$

with

$$M_p(f, r) = \left( \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

In particular,
$L(p, p, 1/p) = L^p(\mathbb{D}, dA), L(p, p, \beta/p) = L^p(\mathbb{D}, (1 - |z|)^{\beta - 1} dA).$

# Weighted mixed norm spaces

# Weighted mixed norm spaces

Let $\rho : (0,1] \to \mathbb{R}^+$ is measurable and bounded on compact sets of $(0,1]$. Define $L(p,q,\rho)$ the space of measurable functions on the unit disc $\mathbb{D}$ such that

$$\|f\|_{L(p,q,\rho)} = \left( \int_0^1 \frac{\rho(1-r)}{1-r} M_p^q(f,r) dr \right)^{1/q} < \infty. \tag{5}$$

# Weighted mixed norm spaces

Let $\rho : (0,1] \to \mathbb{R}^+$ is measurable and bounded on compact sets of $(0,1]$. Define $L(p,q,\rho)$ the space of measurable functions on the unit disc $\mathbb{D}$ such that

$$\|f\|_{L(p,q,\rho)} = \left( \int_0^1 \frac{\rho(1-r)}{1-r} M_p^q(f,r) dr \right)^{1/q} < \infty. \tag{5}$$

In particular, $\rho(t) = t^{\alpha/p}, L(p,q,\rho) = L(p,q,\alpha)$.

## The conditions on weights

For $\varepsilon \in \mathbb{R}$, $\rho$ is said to satisfy **Dini condition of order** $\varepsilon$, in short $\rho \in D_\varepsilon$, if $\int_0^1 \frac{\rho(t)t^\varepsilon}{t}dt < \infty$ and there exist $C > 0$ such that

$$\int_0^s \frac{\rho(t)t^\varepsilon}{t}dt \leq C\rho(s)s^\varepsilon, 0 < s \leq 1. \tag{6}$$

## The conditions on weights

For $\varepsilon \in \mathbb{R}$, $\rho$ is said to satisfy **Dini condition of order** $\varepsilon$, in short $\rho \in D_\varepsilon$, if $\int_0^1 \frac{\rho(t)t^\varepsilon}{t} dt < \infty$ and there exist $C > 0$ such that

$$\int_0^s \frac{\rho(t)t^\varepsilon}{t} dt \leq C\rho(s)s^\varepsilon, 0 < s \leq 1. \tag{6}$$

For $\delta \in \mathbb{R}$, $\rho$ is said to satisfy the $b_\delta$**-condition**, in short $\rho \in b_\delta$, if there exist $C > 0$ such that

$$\int_s^1 \frac{\rho(t)}{t^{1+\delta}} dt \leq C\frac{\rho(s)}{s^\delta}, 0 < s \leq 1. \tag{7}$$

## The conditions on weights

For $\varepsilon \in \mathbb{R}$, $\rho$ is said to satisfy **Dini condition of order** $\varepsilon$, in short $\rho \in D_\varepsilon$, if $\int_0^1 \frac{\rho(t)t^\varepsilon}{t}dt < \infty$ and there exist $C > 0$ such that

$$\int_0^s \frac{\rho(t)t^\varepsilon}{t}dt \leq C\rho(s)s^\varepsilon, 0 < s \leq 1. \tag{6}$$

For $\delta \in \mathbb{R}$, $\rho$ is said to satisfy the $b_\delta$-**condition**, in short $\rho \in b_\delta$, if there exist $C > 0$ such that

$$\int_s^1 \frac{\rho(t)}{t^{1+\delta}}dt \leq C\frac{\rho(s)}{s^\delta}, 0 < s \leq 1. \tag{7}$$

For $\rho(t) = t^\alpha \left(\log(\frac{e}{t})\right)^\beta$, $\rho \in D_\varepsilon \cap b_\delta$ if and only if $-\varepsilon < \alpha < \delta$.

## A basic lemma

### Lemma

*Let $\varepsilon, \delta \in \mathbb{R}$ such that $\varepsilon + \delta \neq 0$. The following are equivalent*

*(i)* $\rho \in D_\varepsilon \cap b_\delta$.

*(ii)* $\int_0^1 \frac{\rho(1-r)(1-r)^{\varepsilon-1}}{(1-rs)^{\varepsilon+\delta}} dr \leq C \frac{\rho(1-s)}{(1-s)^\delta}, 1/2 \leq s < 1.$

## A basic lemma

### Lemma

*Let $\varepsilon, \delta \in \mathbb{R}$ such that $\varepsilon + \delta \neq 0$. The following are equivalent*
*(i)* $\rho \in D_\varepsilon \cap b_\delta$.
*(ii)* $\int_0^1 \frac{\rho(1-r)(1-r)^{\varepsilon-1}}{(1-rs)^{\varepsilon+\delta}} dr \leq C \frac{\rho(1-s)}{(1-s)^\delta}, 1/2 \leq s < 1.$

*Proof* For $1/2 \leq s < 1$

$$\int_0^1 \frac{\rho(1-r)(1-r)^{\varepsilon-1}}{(1-rs)^{\varepsilon+\delta}} dr = \int_0^1 \frac{\rho(t) t^{\varepsilon-1}}{(st+(1-s))^{\varepsilon+\delta}} dt$$

## A basic lemma

### Lemma

*Let $\varepsilon, \delta \in \mathbb{R}$ such that $\varepsilon + \delta \neq 0$. The following are equivalent*
*(i)* $\rho \in D_\varepsilon \cap b_\delta$.
*(ii)* $\int_0^1 \frac{\rho(1-r)(1-r)^{\varepsilon-1}}{(1-rs)^{\varepsilon+\delta}} dr \leq C \frac{\rho(1-s)}{(1-s)^\delta}, 1/2 \leq s < 1.$

*Proof* For $1/2 \leq s < 1$

$$\int_0^1 \frac{\rho(1-r)(1-r)^{\varepsilon-1}}{(1-rs)^{\varepsilon+\delta}} dr = \int_0^1 \frac{\rho(t)t^{\varepsilon-1}}{(st+(1-s))^{\varepsilon+\delta}} dt$$

$$\approx \int_0^{1-s} \frac{\rho(t)t^{\varepsilon-1}}{(t+(1-s))^{\varepsilon+\delta}} dt + \int_{1-s}^1 \frac{\rho(t)t^{\varepsilon-1}}{(t+(1-s))^{\varepsilon+\delta}} dt$$

## A basic lemma

### Lemma

*Let $\varepsilon, \delta \in \mathbb{R}$ such that $\varepsilon + \delta \neq 0$. The following are equivalent*
*(i)* $\rho \in D_\varepsilon \cap b_\delta$.
*(ii)* $\int_0^1 \frac{\rho(1-r)(1-r)^{\varepsilon-1}}{(1-rs)^{\varepsilon+\delta}} dr \leq C \frac{\rho(1-s)}{(1-s)^\delta}, 1/2 \leq s < 1$.

*Proof* For $1/2 \leq s < 1$

$$\int_0^1 \frac{\rho(1-r)(1-r)^{\varepsilon-1}}{(1-rs)^{\varepsilon+\delta}} dr = \int_0^1 \frac{\rho(t)t^{\varepsilon-1}}{(st+(1-s))^{\varepsilon+\delta}} dt$$

$$\approx \int_0^{1-s} \frac{\rho(t)t^{\varepsilon-1}}{(t+(1-s))^{\varepsilon+\delta}} dt + \int_{1-s}^1 \frac{\rho(t)t^{\varepsilon-1}}{(t+(1-s))^{\varepsilon+\delta}} dt$$

$$\approx \frac{1}{(1-s)^{\varepsilon+\delta}} \int_0^{1-s} \frac{\rho(t)t^\varepsilon}{t} dt + \int_{1-s}^1 \frac{\rho(t)}{t^{1+\delta}} dt$$

## The averaging operator

Recall the notation

$$A_\alpha(\varphi)(z) = \frac{1}{|A(z,\alpha)|}\int_{A(z,\alpha)} \varphi(\omega)dA(\omega)$$

and

$$\hat{\varphi}_s(z) = \frac{1}{|D(z,s)|}\int_{D(z,s)} \varphi(\omega)dA(\omega)$$

where $D(z,s) = \{w : |\frac{w-z}{1-\bar{z}w}| < s\}$.

## The averaging operator

Recall the notation

$$A_\alpha(\varphi)(z) = \frac{1}{|A(z,\alpha)|}\int_{A(z,\alpha)} \varphi(\omega)dA(\omega)$$

and

$$\hat{\varphi}_s(z) = \frac{1}{|D(z,s)|}\int_{D(z,s)} \varphi(\omega)dA(\omega)$$

where $D(z,s) = \{w : |\frac{w-z}{1-\bar{z}w}| < s\}$.

## Theorem

*Let $0 < p \leq \infty$ and $\rho$ be a non-negative continuous function. The following are equivalent:*
*(i) $\rho \in b_1$.*
*(ii) The averaging operator $A_\alpha$ is bounded on $L(p,1,\rho)$ for all $1 \leq p \leq \infty$.*
*(iii) The averaging operator $A_\alpha$ is bounded on $L(p,1,\rho)$ for some $0 < p \leq \infty$.*

### Theorem

*Let $1 \leq p \leq \infty$ and $1 < q < \infty$. If $\rho \in b_\delta$ for some $\delta < q$ then the averaging operator $A_\alpha$ is bounded on $L(p,q,\rho)$.*

## Theorem

*Let $1 \leq p \leq \infty$ and $1 < q < \infty$. If $\rho \in b_\delta$ for some $\delta < q$ then the averaging operator $A_\alpha$ is bounded on $L(p, q, \rho)$.*

## Corollary

*Let $1 \leq p, q < \infty$. Then the averaging operator $\varphi \to \widehat{\varphi}_s$ is bounded on $L(p, q, \alpha)$ for $\alpha < 1$.*

## Bergman type projections

Denote $H(p,q,\rho) = L(p,q,\rho) \cap \mathscr{H}(\mathbb{D})$. In particular $H(p,p,\alpha/p) = A^p_\alpha(\mathbb{D})$.

## Bergman type projections

Denote $H(p,q,\rho)=L(p,q,\rho)\cap\mathscr{H}(\mathbb{D})$. In particular $H(p,p,\alpha/p)=A^p_\alpha(\mathbb{D})$.

## Theorem

*Let $\alpha>-1$, $1\le p\le\infty$, $1<q<\infty$, $\rho\in D_\varepsilon\cap b_\delta$ for some $\varepsilon<0$ and $\delta<q(1+\alpha)$ with $\varepsilon+\delta\neq 0$. Then $P_\alpha$ is a projection from $L(p,q,\rho)$ onto $H(p,q,\rho)$.*

## Bergman type projections

Denote $H(p,q,\rho) = L(p,q,\rho) \cap \mathscr{H}(\mathbb{D})$. In particular $H(p,p,\alpha/p) = A^p_\alpha(\mathbb{D})$.

## Theorem

*Let $\alpha > -1$, $1 \leq p \leq \infty$, $1 < q < \infty$, $\rho \in D_\varepsilon \cap b_\delta$ for some $\varepsilon < 0$ and $\delta < q(1+\alpha)$ with $\varepsilon + \delta \neq 0$. Then $P_\alpha$ is a projection from $L(p,q,\rho)$ onto $H(p,q,\rho)$.*

## Corollary

*Let $1 \leq p, q < \infty$, $\alpha, \gamma > -1$ and $\beta \in \mathbb{R}$.*
*(i) Then $P_\alpha$ is bounded on $L(p,q,\beta)$ whenever $0 < \beta < (1+\alpha)$ (see [J]).*
*In particular $P_\alpha$ is bounded on $L^p(\mathbb{D},(1-|z|^2)^\gamma dA(z))$ whenever $0 < \gamma + 1 < p(1+\alpha)$ (see [HKZ]).*

## Berezin transform and mixed norm spaces

If $T : A^2 \to A^2$ is bounded and $k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}$ is the normalized Bergman kernel then

$$\tilde{T}(z) = \langle T(k_z), k_z \rangle.$$

## Berezin transform and mixed norm spaces

If $T : A^2 \to A^2$ is bounded and $k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}$ is the normalized Bergman kernel then

$$\tilde{T}(z) = \langle T(k_z), k_z \rangle.$$

For Toeplitz operators

$$T_\varphi(f)(z) = P_0(\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\bar{w}z)^2} dA(w)$$

one has that $\widetilde{\varphi}(z) = \widetilde{T_\varphi}(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\varphi(w)}{|1-\bar{w}z|^4} dA(w)$.

## Berezin transform and mixed norm spaces

If $T : A^2 \to A^2$ is bounded and $k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}$ is the normalized Bergman kernel then

$$\tilde{T}(z) = \langle T(k_z), k_z \rangle.$$

For Toeplitz operators

$$T_\varphi(f)(z) = P_0(\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1 - \bar{w}z)^2} dA(w)$$

one has that $\widetilde{\varphi}(z) = \widetilde{T_\varphi}(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{\varphi(w)}{|1-\bar{w}z|^4} dA(w)$.
Therefore, for $1 \le p < \infty$

$$|\widetilde{\varphi}(z)|^p \le (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{|\varphi(w)|^p}{|1 - \bar{w}z|^4} dA(w),$$

## Berezin transform and mixed norm spaces

If $T : A^2 \to A^2$ is bounded and $k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}$ is the normalized Bergman kernel then

$$\tilde{T}(z) = \langle T(k_z), k_z \rangle.$$

For Toeplitz operators

$$T_\varphi(f)(z) = P_0(\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\bar{w}z)^2} dA(w)$$

one has that $\widetilde{\varphi}(z) = \widetilde{T_\varphi}(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\varphi(w)}{|1-\bar{w}z|^4} dA(w)$.
Therefore, for $1 \le p < \infty$

$$|\widetilde{\varphi}(z)|^p \le (1-|z|^2)^2 \int_{\mathbb{D}} \frac{|\varphi(w)|^p}{|1-\bar{w}z|^4} dA(w),$$

Hence if $\phi \in L^p(\mathbb{D}, \frac{dA(z)}{(1-|z|^2)^2})$ then $\widetilde{\phi} \in L^p(\mathbb{D}, \frac{dA(z)}{(1-|z|^2)^2})$ for $1 \le p < \infty$.

## Berezin transform and mixed norm spaces

If $T : A^2 \to A^2$ is bounded and $k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}$ is the normalized Bergman kernel then

$$\tilde{T}(z) = \langle T(k_z), k_z \rangle.$$

For Toeplitz operators

$$T_\varphi(f)(z) = P_0(\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\bar{w}z)^2} dA(w)$$

one has that $\widetilde{\varphi}(z) = \widetilde{T_\varphi}(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\varphi(w)}{|1-\bar{w}z|^4} dA(w)$.
Therefore, for $1 \leq p < \infty$

$$|\widetilde{\varphi}(z)|^p \leq (1-|z|^2)^2 \int_{\mathbb{D}} \frac{|\varphi(w)|^p}{|1-\bar{w}z|^4} dA(w),$$

Hence if $\phi \in L^p(\mathbb{D}, \frac{dA(z)}{(1-|z|^2)^2})$ then $\widetilde{\phi} \in L^p(\mathbb{D}, \frac{dA(z)}{(1-|z|^2)^2})$ for $1 \leq p < \infty$.

## Theorem

*(B-Pérez-Esteva (IEOT-2011)) Let* $1 \leq p, q < \infty$. *Then the Berezin transform is bounded on* $L(p, q, \alpha)$ *for* $-2 < \alpha < 1$.

**Theorem**

*(B-Pérez-Esteva (IEOT-2011)) Let* $1 \leq p, q < \infty$. *Then the Berezin transform is bounded on* $L(p,q,\alpha)$ *for* $-2 < \alpha < 1$.

**Theorem**

*Let* $\rho$ *non negative continuous function. The following are equivalent:*
*(i)* $\rho \in D_2 \cap b_1$, *i.e.* $\int_0^s \rho(t) t dt \leq C s^2 \rho(s), \int_s^1 \frac{\rho(t)}{t^2} dt \leq C \frac{\rho(s)}{s}$.
*(ii) The Berezin transform is bounded on* $L(p,1,\rho)$ *for all* $1 \leq p \leq \infty$.
*(iii) The Berezin transform is bounded on* $L(p,1,\rho)$ *for some* $0 < p \leq \infty$.

### Theorem

*(B-Pérez-Esteva (IEOT-2011)) Let* $1 \leq p, q < \infty$*. Then the Berezin transform is bounded on* $L(p, q, \alpha)$ *for* $-2 < \alpha < 1$*.*

### Theorem

*Let* $\rho$ *non negative continuous function. The following are equivalent:*
*(i)* $\rho \in D_2 \cap b_1$*, i.e.* $\int_0^s \rho(t) t dt \leq C s^2 \rho(s), \int_s^1 \frac{\rho(t)}{t^2} dt \leq C \frac{\rho(s)}{s}$*.*
*(ii) The Berezin transform is bounded on* $L(p, 1, \rho)$ *for all* $1 \leq p \leq \infty$*.*
*(iii) The Berezin transform is bounded on* $L(p, 1, \rho)$ *for some* $0 < p \leq \infty$*.*

### Theorem

*Let* $1 \leq p \leq \infty$ *and* $1 < q < \infty$*.*
*If* $\rho \in D_\varepsilon \cap b_\delta$ *for some* $\varepsilon < 2q$ *and* $\delta < q$ *with* $\varepsilon + \delta \neq 0$ *then the Berezin transform is bounded on* $L(p, q, \rho)$*.*

## From conditions on weights to boundedness of Berezin transform

We show that $\rho \in D_2 \cap b_1$ implies that the Berezin transform is bounded on $L(p,1,\rho)$ for all $1 \leq p \leq \infty$.

## From conditions on weights to boundedness of Berezin transform

We show that $\rho \in D_2 \cap b_1$ implies that the Berezin transform is bounded on $L(p,1,\rho)$ for all $1 \leq p \leq \infty$.Let $f \in L(p,1,\rho)$.
Use Minkoswki's and standard estimates to obtain

$$M_p(\tilde{f},r) \leq C(1-r)^2 \int_0^1 \frac{M_p(f,s)}{(1-rs)^3} ds. \tag{8}$$

## From conditions on weights to boundedness of Berezin transform

We show that $\rho \in D_2 \cap b_1$ implies that the Berezin transform is bounded on $L(p,1,\rho)$ for all $1 \leq p \leq \infty$.Let $f \in L(p,1,\rho)$.
Use Minkoswki's and standard estimates to obtain

$$M_p(\tilde{f},r) \leq C(1-r)^2 \int_0^1 \frac{M_p(f,s)}{(1-rs)^3} ds. \tag{8}$$

Now

$$\begin{aligned}\int_0^1 \frac{\rho(1-r)}{1-r} M_p(\tilde{f},r)dr &\leq \int_0^1 \rho(1-r)(1-r)(\int_0^1 \frac{M_p(f,s)}{(1-rs)^3} ds)dr \\ &\leq C\int_0^1 (\int_0^1 \frac{\rho(1-r)(1-r)}{(1-rs)^3} dr)M_p(f,s)ds.\end{aligned}$$

## From conditions on weights to boundedness of Berezin transform

We show that $\rho \in D_2 \cap b_1$ implies that the Berezin transform is bounded on $L(p,1,\rho)$ for all $1 \leq p \leq \infty$.Let $f \in L(p,1,\rho)$.
Use Minkoswki's and standard estimates to obtain

$$M_p(\tilde{f},r) \leq C(1-r)^2 \int_0^1 \frac{M_p(f,s)}{(1-rs)^3} ds. \tag{8}$$

Now

$$\begin{aligned}\int_0^1 \frac{\rho(1-r)}{1-r} M_p(\tilde{f},r)dr &\leq \int_0^1 \rho(1-r)(1-r)(\int_0^1 \frac{M_p(f,s)}{(1-rs)^3} ds)dr \\ &\leq C\int_0^1 (\int_0^1 \frac{\rho(1-r)(1-r)}{(1-rs)^3} dr)M_p(f,s)ds.\end{aligned}$$

The basic lemma applied for $\varepsilon = 2$ and $\delta = 1$ gives

$$\int_0^1 \frac{\rho(1-r)}{1-r} M_p(\tilde{f},r)dr \leq C\int_0^1 \frac{\rho(1-s)}{1-s} M_p(f,s)ds.$$

## From boundedness of Berezin transform to conditions on weights

Now we show that the boundedness of Berezin transform on $L(p,1,\rho)$ for some $1\leq p\leq\infty$ implies $\rho\in D_2\cap b_1$.
Apply the assumption to radial positive functions $\varphi$ to obtain

$$\begin{aligned}\int_0^1\frac{\rho(1-r)}{1-r}\widetilde{\varphi}(r)dr &\approx \int_0^1\Big(\int_0^1\frac{(1-r)\rho(1-r)}{(1-rs)^3}dr\Big)\varphi(s)ds\\ &\leq C\int_0^1\frac{\rho(1-s)}{1-s}\varphi(s)ds.\end{aligned}$$

## From boundedness of Berezin transform to conditions on weights

Now we show that the boundedness of Berezin transform on $L(p,1,\rho)$ for some $1 \leq p \leq \infty$ implies $\rho \in D_2 \cap b_1$.
Apply the assumption to radial positive functions $\varphi$ to obtain

$$\begin{aligned}\int_0^1 \frac{\rho(1-r)}{1-r}\widetilde{\varphi}(r)dr &\approx \int_0^1 \left(\int_0^1 \frac{(1-r)\rho(1-r)}{(1-rs)^3}dr\right)\varphi(s)ds \\ &\leq C\int_0^1 \frac{\rho(1-s)}{1-s}\varphi(s)ds.\end{aligned}$$

Therefore

$$\int_0^1 \left(C\frac{\rho(1-s)}{1-s} - \int_0^1 \frac{(1-r)\rho(1-r)}{(1-rs)^3}dr\right)\varphi(s)ds \geq 0$$

for any measurable non-negative function $\varphi$. Hence

$$C\frac{\rho(1-s)}{1-s} - \int_0^1 \frac{(1-r)\rho(1-r)}{(1-rs)^3}dr \geq 0$$

which implies the result from the basic lemma again.