

Complex and Harmonic Analysis 2011

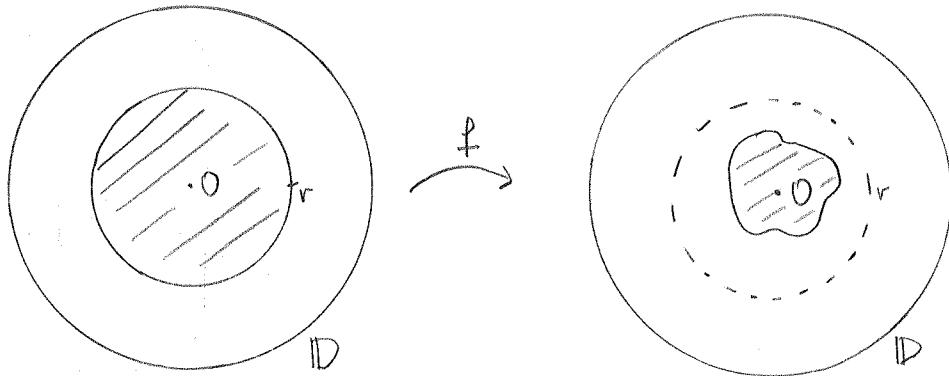
Malaga, 10-14 July 2011

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Variations of Schwarz lemma

The classical Schwarz lemma

$f : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic, $f(0) = 0$.



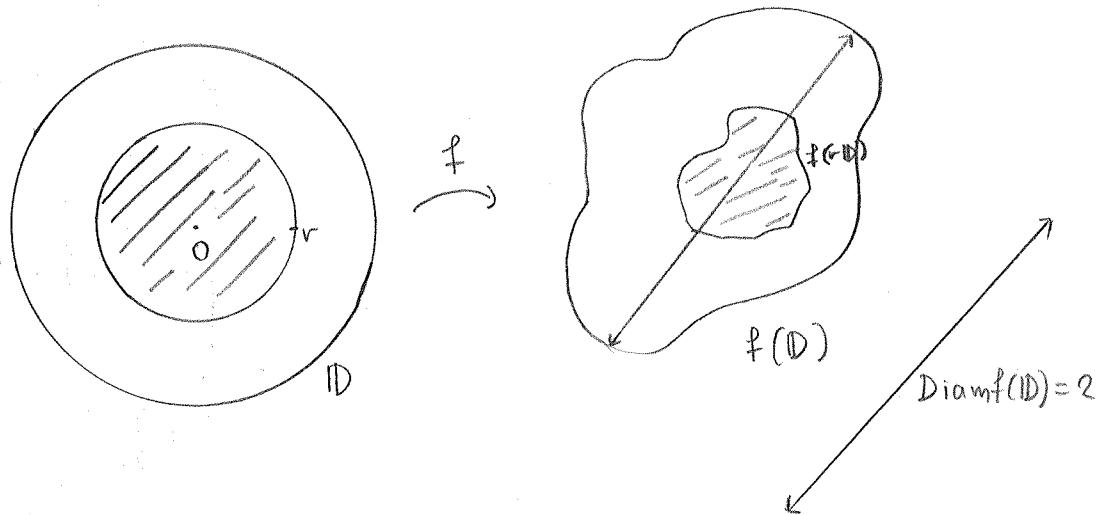
- (a) $|f| \leq 1 \implies |f(z)| \leq |z|, z \in \mathbb{D}$.
- (b) $f(\mathbb{D}) \subset \mathbb{D} \implies f(r\mathbb{D}) \subset r\mathbb{D}, 0 < r < 1$.
- (c) The function

$$\Phi(r) = \frac{\max_{z \in r\mathbb{D}} |f(z)|}{r}, \quad 0 < r < 1,$$

is increasing.

Diameter Schwarz lemma

$f : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic, $f(0) = 0$.



Landau, Toeplitz 1907:

$$\text{Diam } f(\mathbb{D}) = 2 \implies \text{Diam } f(r\mathbb{D}) \leq 2r.$$

Burckel, Marshall, Minda, Poggi-Corradini, Ransford 2008:

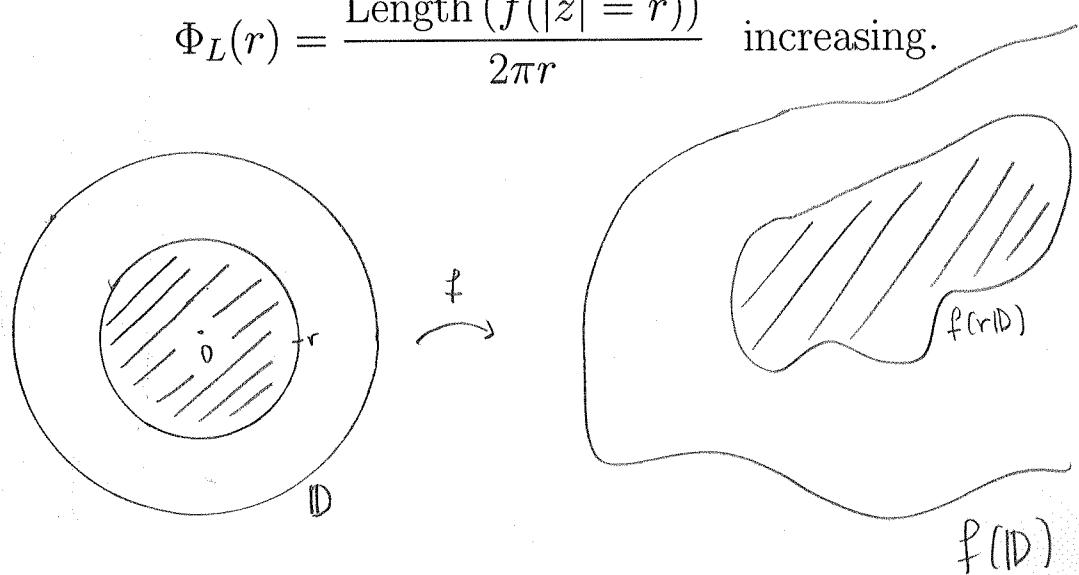
$$\Phi_D(r) = \frac{\text{Diam } f(r\mathbb{D})}{2r} \quad \text{increasing}$$

$$\text{Diam } f(\mathbb{D}) = 2 \implies |f(z)| \leq \frac{2}{1 + \sqrt{1 - |z|^2}} |z|, \quad z \in \mathbb{D}.$$

Length Schwarz lemma

Pólya-Szegö 1924: If, in addition, f is univalent, then

$$\Phi_L(r) = \frac{\text{Length } (f(|z| = r))}{2\pi r} \text{ increasing.}$$



Area Schwarz lemma

Burckel, Marshall, Minda, Poggi-Corradini, Ransford 2008:

$$\Phi_A(r) = \frac{\text{Area } f(r\mathbb{D})}{\pi r^2} \text{ increasing}$$

Related results:

Dubinin 2011,

Betsakos 2010 for quasiregular maps.

Capacity Schwarz lemmas

Burckel, Marshall, Minda, Poggi-Corradini, Ransford 2008
for logarithmic capacity:

$$\Phi_C(r) = \frac{\text{Cap } f(r\mathbb{D})}{r} \quad \text{increasing}$$

Betsakos, Pouliasis 2011
for hyperbolic capacity and $f : \mathbb{D} \rightarrow \mathbb{D}$,

$$\Phi_C(r) = \frac{\text{Cap}_h f(r\mathbb{D})}{r} \quad \text{increasing}$$

Eigenvalue Schwarz lemma

Laugesen, Morpurgo 1998 (f univalent):

$$\Phi_\lambda(r) = \frac{\lambda(f(r\mathbb{D}))^{-1}}{r^2} \quad \text{increasing}$$

Hardy norm Schwarz lemma

$f : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic, $f(0) = 0$, $0 < p < \infty$.

Julia 1927:

$$\Phi_H(r) = \frac{\int_0^{2\pi} |f(re^{i\theta})|^p d\theta}{r^p} \quad \text{increasing}$$

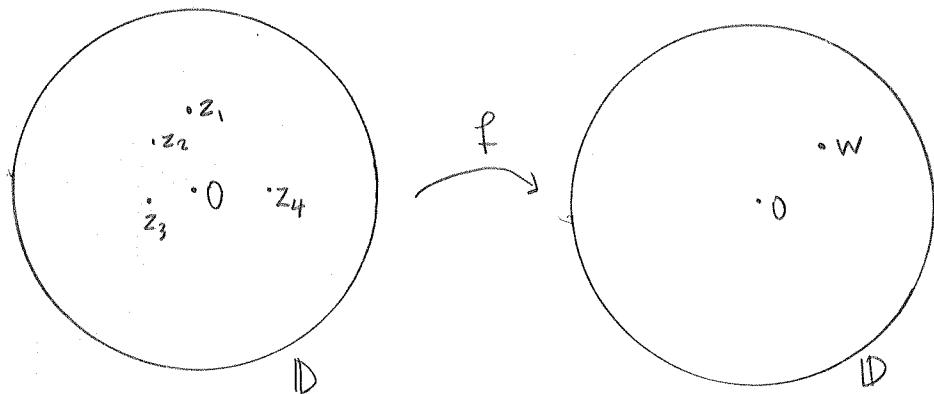
Bergman norm Schwarz lemma

$f : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic, $0 < p < \infty$.

Xiao, Zhu 2011:

$$\Phi_B(r) = \frac{\int_{r\mathbb{D}} |f|^p dA}{\pi r^2} \quad \text{increasing}$$

Littlewood's theorem



$f : \mathbb{D} \rightarrow \mathbb{D}$ holomorphic, $f(0) = 0$, $w \in f(\mathbb{D}) \setminus \{0\}$,
 $f(z_j(w)) = w$.

Jensen 1919, Littlewood 1925:

$$|w| \leq \prod_j |z_j(w)|$$

Lehto 1954: Equality $\iff f$ inner function.

Equivalent inequality:

$$N(w, f) = \sum_j \log \frac{1}{|z_j(w)|} \leq \log \frac{1}{|w|}.$$

Theorem

$f : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic, $f(0) = 0$,

$\text{Diam } f(\mathbb{D}) = 2$,

$w \in f(\mathbb{D})$, $f(z_j(w)) = w$.

Then

$$\frac{4|w|}{4 + |w|^2} \leq \prod_j |z_j(w)|.$$

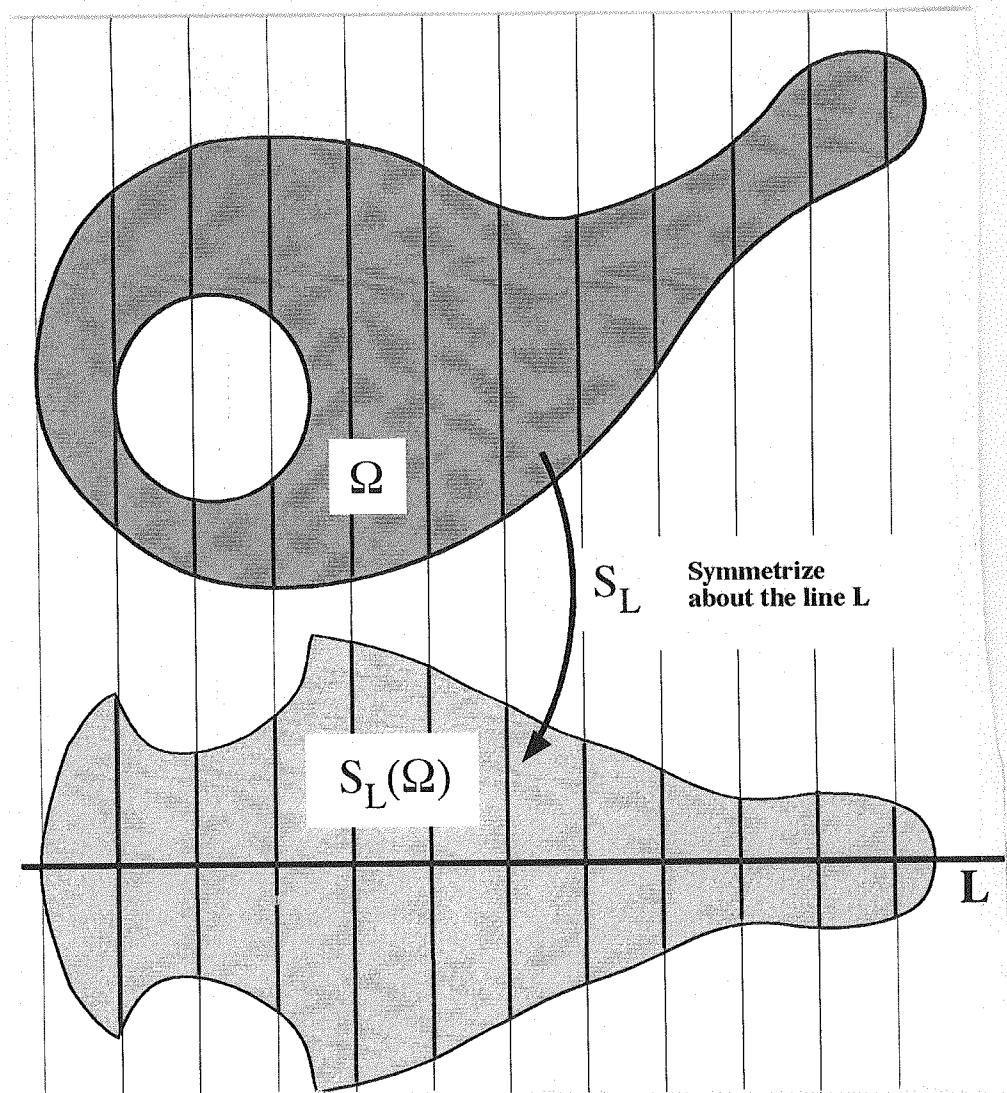
Equality $\iff f(z) = \phi \circ h$,

where ϕ is a linear fractional transformation and h is inner.

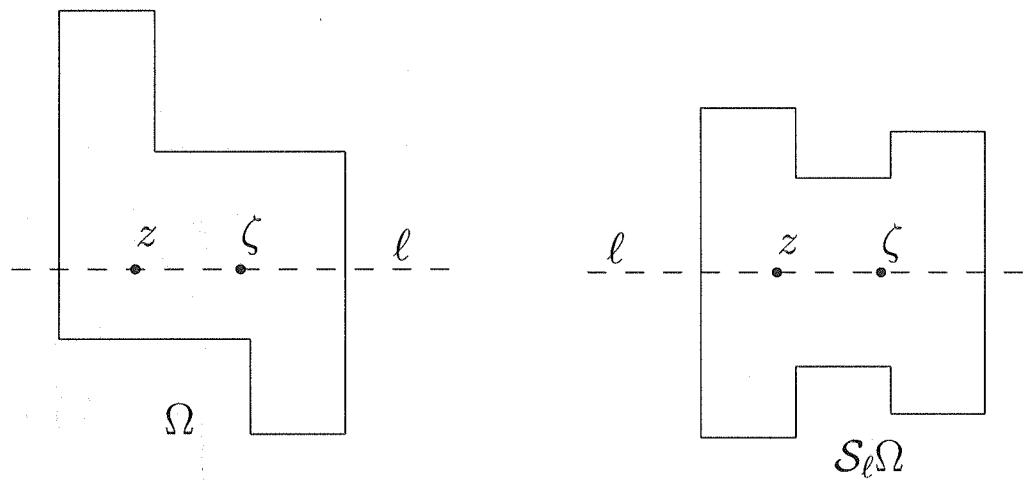
Burckel, Marshall, Minda, Poggi-Corradini, Ransford:

$$\frac{4|w|}{4 + |w|^2} \leq |z|$$

Steiner symmetrization



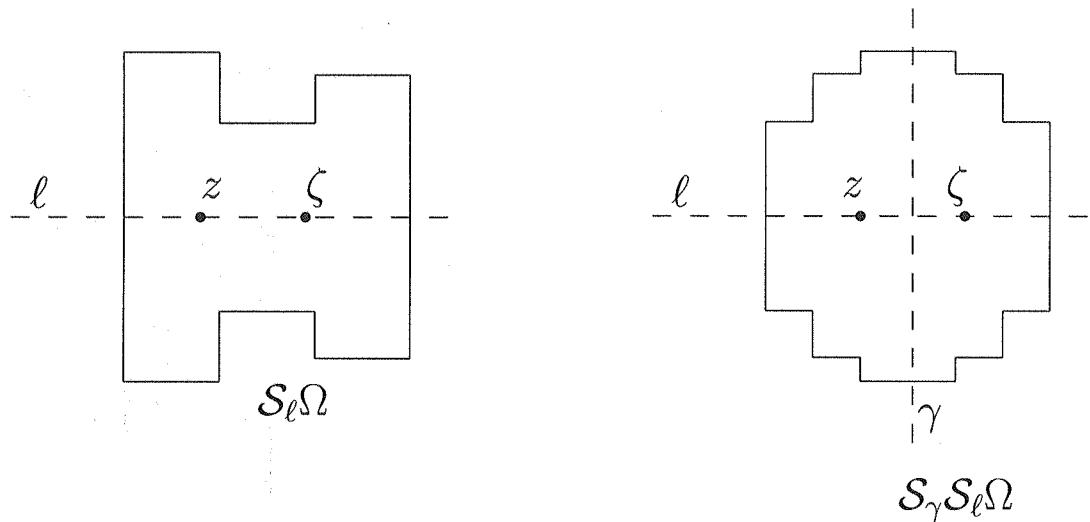
Steiner symmetrization and Green function



Baernstein 1974:

$$g(z, \zeta, \Omega) \leq g(z, \zeta, S_\ell \Omega)$$

Lemma 1



$$g(z, \zeta, \mathcal{S}_\ell \Omega) \leq g(z, \zeta, \mathcal{S}_\gamma \mathcal{S}_\ell \Omega)$$

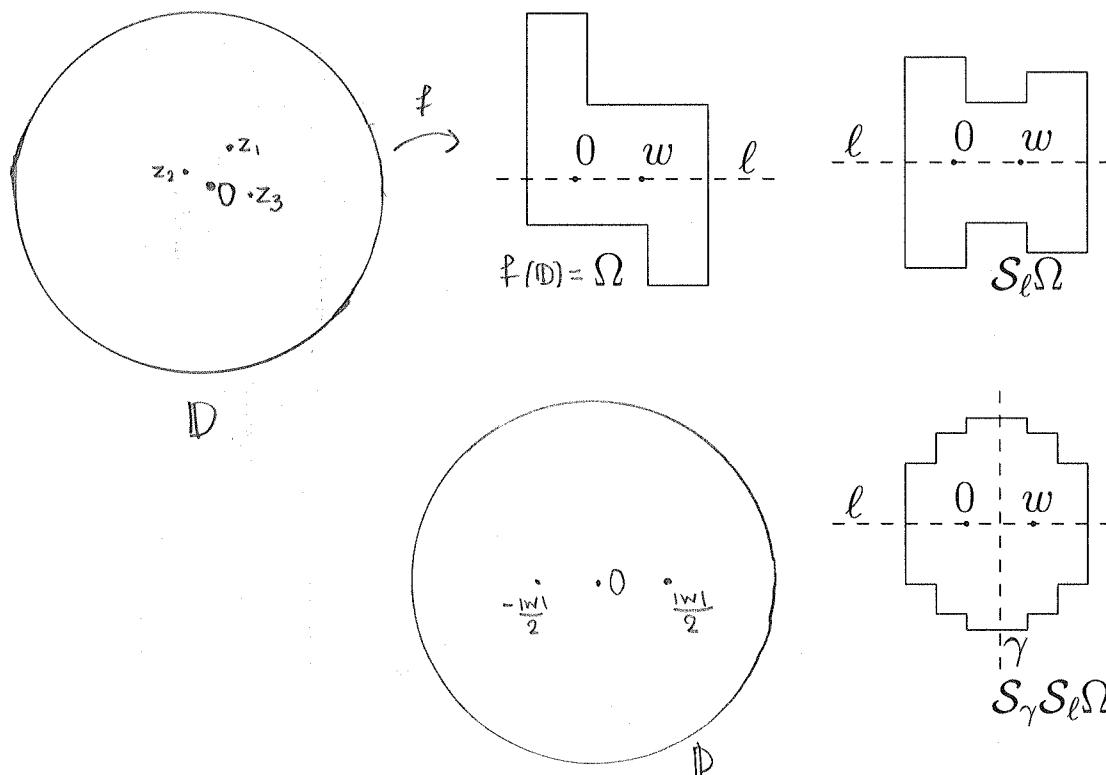
Proof of the theorem

Theorem: $f : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic, $f(0) = 0$,
 $\text{Diam } f(\mathbb{D}) = 2$, $w \in f(\mathbb{D})$, $f(z_j(w)) = w$.

Then

$$\frac{4|w|}{4 + |w|^2} \leq \prod_j |z_j(w)|.$$

Proof



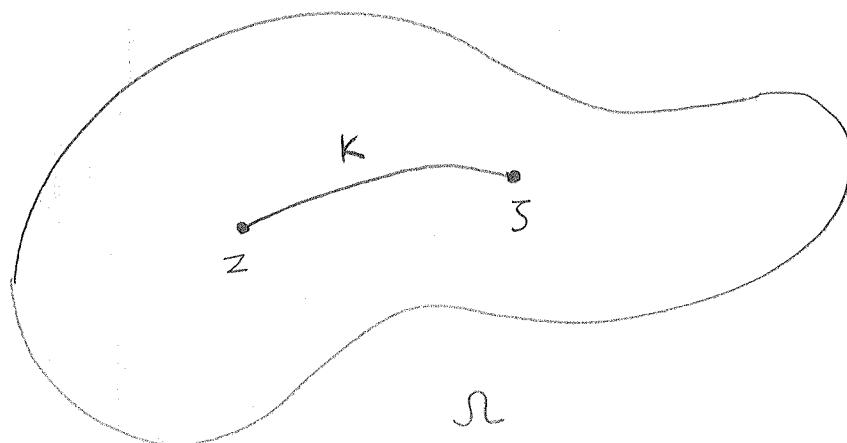
$$\begin{aligned}
 \sum_j \log \frac{1}{|z_j(w)|} &= \sum_j g(0, z_j(w), \mathbb{D}) \leq g(0, w, \Omega) \\
 &\leq g(0, w, \mathcal{S}_\ell \Omega) \leq g(0, w, \mathcal{S}_\gamma \mathcal{S}_\ell \Omega) \\
 &\leq g(-|w|/2, |w|/2, \mathbb{D}) = \log \frac{4 + |w|^2}{4|w|}.
 \end{aligned}$$

The modulus metric

For a domain Ω , let

$$\mu(z, \zeta, \Omega) = \inf_{z, \zeta \in K} \text{cap}(\Omega, K).$$

See Vuorinen's 1988 book.



Proof of Lemma 1



$$g(z, \zeta, S_\ell\Omega) \leq g(z, \zeta, S_\gamma S_\ell\Omega)$$

Proof

It suffices to prove that

$$\mu(z, \zeta, S_\ell\Omega) \geq \mu(z, \zeta, S_\gamma S_\ell\Omega).$$

This is true because

$$\mu(z, \zeta, S_\ell\Omega) = \text{cap}(S_\ell\Omega, [z, \zeta])$$

$$\geq \text{cap}(S_\gamma S_\ell\Omega, S_\gamma[z, \zeta])$$

$$= \text{cap}(S_\gamma S_\ell\Omega, [z, \zeta])$$

$$= \mu(z, \zeta, S_\gamma S_\ell\Omega).$$

Proof of the area Schwarz lemma

$$\Phi_A(r) = \frac{\text{Area } f(r\mathbb{D})}{\pi r^2} \quad \text{increasing}$$

Proof

Schwarz symmetrization reduces the capacity of condensers:

Let $0 < r < s < 1$. Then

$$\begin{aligned} 2\pi \left(\log \frac{s}{r} \right)^{-1} &= \text{cap}(s\mathbb{D}, \overline{r\mathbb{D}}) \\ &\geq \text{cap}(f(s\mathbb{D}), f(\overline{r\mathbb{D}})) \\ &\geq \text{cap}(s^\sharp\mathbb{D}, \overline{r^\sharp\mathbb{D}}) \\ &= 2\pi \left(\log \frac{s^\sharp}{r^\sharp} \right)^{-1}. \end{aligned}$$

Hence

$$\frac{r^\sharp}{r} \leq \frac{s^\sharp}{s}$$

which gives

$$\frac{\pi(r^\sharp)^2}{\pi r^2} \leq \frac{\pi(s^\sharp)^2}{\pi s^2}$$

or

$$\Phi_A(r) \leq \Phi_A(s).$$

Quiz

How many articles in the 21st century have “Schwarz lemma” in their title?

Answer

At least 50