Inner functions in Möbius invariant spaces

Fernando Pérez-González (U. La Laguna)
and
Jouni Rättyä (U. Eastern Finland-Joensuu)

CHARM 2011, Málaga

Introduction

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It is well known that every such function can be represented as a product of a Blaschke product and a singular inner function.

For a given sequence $\{z_n\}$ in $\mathbb{D}$ for which $\sum_{n=1}^{\infty}(1-|z_n|^2)$ converges, the *Blaschke product* associated with the sequence $\{z_n\}$ is defined as

$$B(z) := \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}.$$

A *singular inner function* is of the form

$$S(z) := \exp\left(\int_{\mathbb{T}} \frac{z+w}{z-w}\, d\sigma(w)\right) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\right),$$

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where the measure σ on $\mathbb{T}$ is singular with respect to the Lebesgue measure.

If the measure σ is atomic and consists of a point mass concentrated in $w \in \mathbb{T}$, then S is of the form

$$S_{\gamma,w}(z) := \exp\left(\gamma \frac{z+w}{z-w}\right), \qquad 0 < \gamma < \infty.$$

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For weighted Bergman spaces we mention the papers by

- Ahern (1979).
- Girela-Peláez (2006).
- Girela-Peláez-Vukotic (2007).
- Kim (1984).

Other authors who have worked on the topic are:

- Bishop (1990) (1993).
- Danikas and Mouratides (2000).
- Essén and Xiao (1997),
- Girela and González (2001).
- Kutbi (2001).
- Smith (1998).
- Stephenson (1988).
- Verbitskii (1985).
- Vinogradov (1997).

Very recently, D. Girela, C. González and M. Jevtić (2011) have studied conditions for an inner function belong to some spaces of analytic functions such as

- Lipschitz classes,
- Besov spaces,
- Sobolev spaces.

Besov-type spaces

Besov-type spaces B_s^p, $p > 0$ and $s \geq 0$, consists of those analytic functions f in $\mathbb{D}$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}g^s(z,a)\,dA(z) < \infty,$$

where $g(z,a) := -\log|\varphi_a(z)|$ is the Green's function of $\mathbb{D}$ and $\varphi_a(z) := (a-z)/(1-\bar{a}z)$.

Note that the Besov-type space B_s^p is **Möbius invariant**.

The closure of polynomials in B_s^p is the *little Besov-type space* $B_{s,0}^p$, and it consists of those analytic functions f in $\mathbb{D}$ for which

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}g^s(z,a)\,dA(z) = 0\,.$$

The space B^p_0 is the *classical Besov space* B_p which contains no other inner functions than finite Blaschke products (Donaire-Girela-Vukotic (2002), Kim (1984)).

The space B_0^p is the *classical Besov space* B_p which contains no other inner functions than finite Blaschke products (Donaire-Girela-Vukotic (2002), Kim (1984)).

For $0 < s < 1$, the space B_s^2 is the Q_s-space where the only inner functions are Blaschke products whose zeros $\{z_n\}$ have the density $\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^s < \infty$ (Essén- Xiao (1997)).

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However, an application of the Garsia norm in *BMOA* shows that the only inner functions in $B_{1,0}^2 = VMOA$ (the space of analytic functions of *vanishing mean oscillation*) are finite Blaschke products.

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Therefore, our range of interest is $0 < s < 1$

Theorem 1

Theorem

Let $0 < p < \infty$ *and* $0 \leq s < 1$ *such that* $p + s > 1$. *Then* $S_{\gamma,w} \notin B_s^p$ *and* $S_{\gamma,w} \notin B_{1,0}^p$, *but* $S_{\gamma,w} \in B_1^p$.

The following result on radial integrability of inner functions plays an important role in some of the proofs. It also generalizes results of Ahern and Clark (1976), Essén and Xiao (1997) and Verbitskii (1985).

Lemma (Radial Variation)

Let S be an inner function and let $1 \le p < \infty$ and $-1 < q < \infty$ such that $p > q + 1$. Then, for any $0 \le \delta < 1$, there is a constant C, depending only on p and q, such that

$$C \int_\delta^1 (1 - |S(re^{i\theta})|^2)^p (1 - r^2)^{q-p}\, dr \quad \le \int_\delta^1 |S'(re^{i\theta})|^p (1 - r^2)^q\, dr$$

$$\le \int_\delta^1 (1 - |S(re^{i\theta})|^2)^p (1 - r^2)^{q-p}\, dr$$

for almost all θ in $[0, 2\pi)$.

Theorem 2

The second of the main results generalizes in part Theorem 1 to the case when the generating measure σ of the singular inner function S is non-atomic.

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Let $0 < p < \infty$ and $0 \leq s < 1$ such that $p + s > 1$. Then B_s^p does not contain any singular inner functions.

According to the previous results, the only possible inner functions in B_s^p, $0 \leq s < 1$, and $B_{1,0}^p$ are Blaschke products.

As mentioned earlier, the only inner functions in the classical Besov space $B_p = B_0^p$ are finite Blaschke products.

Something more can be said for B_s^p spaces, if $0 < s \leq 1$ and $p > 1 - s$.

Theorem 3

Theorem

Let $0 < s < 1$. Then an inner function belongs to the Möbius invariant Besov-type space B_s^p for some (equivalently for all) $p > \text{máx}\{s, 1-s\}$ if and only if it is the Blaschke product associated with a sequence $\{z_n\}$ which satisfies

$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^s < \infty. \tag{4.1}$$

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This result can also be considered as a refinement of the known fact that an inner function belongs to $Q_s = B^2_s$, $0 < s < 1$, if and only if it is a Blaschke product whose zeros $\{z_n\}$ satisfy $\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^s < \infty$ (Essén and Xiao (1997) and Verbitskii (1985)).

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$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^s < \infty.$$

For $0 < s \leq \frac{1}{2}$, the assertion in theorem is sharp in the sense that $1 - s \geq s$ and the condition $p > 1 - s$ only guarantees that the space B_s^p is not trivial.

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$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^s < \infty.$$

The assertion in theorem clearly fails for $s = 1$ since $B_1^2 = BMOA$ contains all bounded analytic functions while the condition for $s = 1$ is satisfied if and only if $\{z_n\}$ is a finite union of uniformly separated sequences.

Recall that a sequence $\{z_n\} \subset \mathbb{D}$ is called *uniformly separated*, if there exists a δ such that

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A famous result by Carleson states that $\{z_n\}$ is an interpolating sequence for the space H^∞ of all bounded analytic functions in $\mathbb{D}$ if and only if it is uniformly separated.

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A famous result by Carleson states that $\{z_n\}$ is an interpolating sequence for the space H^∞ of all bounded analytic functions in $\mathbb{D}$ if and only if it is uniformly separated.

It is also worth observing that (4.1) for $s = 2$ is satisfied if and only if $\{z_n\}$ is a finite union of uniformly discrete sequences. A sequence $\{z_n\} \subset \mathbb{D}$ is called *uniformly discrete* (or *separated*), if there exists a δ such that

$$\inf_{k \neq j} |\varphi_{z_k}(z_j)| \geq \delta > 0.$$

(Duren-Schuster-Vukotić (2005)).

For $\xi \in \mathbb{T}$ and $M \in [1, \infty)$, the domain

$$\{z \in \mathbb{D} : |1 - \overline{\xi} z| \leq M(1 - |z|^2)\}$$

is called a *Stolz angle* with vertex at ξ.

Blaschke products whose zeros lie in such an angular domain and which belong to Q_s have been studied by Danikas and Mouratides (2000). Theorem 3 and the proofs of Theorems 1 and 2 in the Danikas-Mouratides' paper yield corollary.

Corollary 4

Corollary

Let $0 < s < 1$ *and* $p > \max\{s, 1-s\}$, *and let* B *be a Blaschke product with zeros* $\{z_n\}$ *in a Stolz angle. Then* $B \in B_s^p$ *if and only if* $\{1 - |z_n|\}$ *is NOT asymptotically concentrated.*

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Let $0 < s < 1$ and $p > \text{máx}\{s, 1-s\}$, and let B be a Blaschke product with zeros $\{z_n\}$ in a Stolz angle. Then $B \in B_s^p$ if and only if $\{1 - |z_n|\}$ is NOT asymptotically concentrated.

A decreasing sequence $\{a_n\}$ of positive real numbers tending to zero is called *asymptotically concentrated*, if for any $k \in \mathbb{N} := \{1, 2, \dots\}$ there is an infinite sequence $\{n_j\} \subset \mathbb{N}$, depending on k, such that $(a_{n_j}/a_{n_j+k}) \to 1$ as $j \to \infty$.

Corollary 5

Theorem 3, combined with a result of Reséndis and Tovar (1999), gives a sufficient condition for the zeros of a Blaschke product such that it belongs to B_s^p.

Corollary

Let $0 < s < 1$, and let B be the Blaschke product associated with a sequence $\{z_n\}$. If there exists a positive constant C such that

$$\sum_{n=k+1}^{\infty} (1 - |z_n|^2)^s \leq C(1 - |z_k|^2)^s$$

for all $k \in \mathbb{N}$, then $B \in B_s^p$ for all $p > \text{máx}\{s, 1 - s\}$.

Converse implication is not true in general by results of Reséndis and Tovar.

Theorem 3.1

We use, for Besov-type spaces, a partial improvement of a result due to Vinogradov (1997).

Theorem

Let B be the Blaschke product associated with a sequence $\{z_n\}$.
(a) *If* $0 < p < 1$ *and*

$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^p < \infty,$$

then $B \in \bigcap_{s>\text{máx}\{p,1-p\}} B_s^p$.

Theorem 3.1

Theorem

(b) *If* $\frac{1}{2} < p \leq 1$ *and*

$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^p \log \frac{1}{1 - |\varphi_a(z_n)|^2} < \infty,$$

then $B \in B_p^p$.
(c) *If* $0 < s < 1$ *and*

$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^s < \infty, \tag{4.2}$$

then $B \in \bigcap_{p>\text{máx}\{s,1-s\}} B_s^p$.

Theorem 3.2

The next result is a partial converse of part c) in Theorem 3.1. It uses ideas from Essén and Xiao (1997) and Verbitskii (1995).

Theorem

Let $0 < p < \infty$ and $0 < s < 1$ such that $p + s > 1$, and let B be the Blaschke product associated with a sequence $\{z_n\}$. If $B \in B_s^p$, then (4.2) is satisfied.

Proof of Theorem 3

Let S be an inner function and let $0 < s < 1$. If $S \in B_s^p$ for some $p > 0$, we already know that S have to be a Blaschke product. Moreover, (Thm. 3.2) its zero sequence satisfies

$$sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty}(1 - |\varphi_a(z_n)|^2)^s < \infty \, .$$

Conversely, If $\{z_n\}$ is the zero sequence of a Blaschke product B satisfying the estimation above, then (Thm. 3.1) $B \in B_s^p$ for all $p > \text{máx}\{s, 1 - s\}$.

$s = 1$

The case $s = 1$ was excluded in Theorem 3. Since $B_1^2 = BMOA$ and $B_{1,0}^2 = VMOA$, and the inclusions

$$B_s^{p_1} \subsetneq B_s^{p_2} \quad \text{and} \quad B_{s,0}^{p_1} \subsetneq B_{s,0}^{p_2} \tag{4.3}$$

for all $1 - s < p_1 < p_2 < \infty$ and $0 \leq s \leq 1$, ensure that B_1^p contains all inner functions if $p \geq 2$, and $B_{1,0}^p$ contains no inner functions other than finite Blaschke products when $p \leq 2$.

By Theorem 1, the singular inner function $S_{\gamma,w}$ belongs to B_1^p for all $p > 0$, and does not belong to $B_{1,0}^p$ for any $p > 0$.

Moreover, Theorem 3.1 (b) shows that if the zero-sequence $\{z_n\}$ of a Blaschke product B satisfies

$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1-|\varphi_a(z_n)|^2) \log \frac{1}{1-|\varphi_a(z_n)|^2} < \infty, \tag{4.4}$$

then $B \in B_1^1$.

Conversely, if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B'(z)|(1-|z|^2)^{-1} \log \frac{1}{1-|z|} (1-|\varphi_a(z)|^2)\, dA(z) < \infty, \tag{4.5}$$

then the zero-sequence $\{z_n\}$ of B satisfies (4.4).

If $B^1_{1,\log}$ denotes the space of all analytic functions in $\mathbb{D}$ satisfying (4.5), i.e.

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B'(z)|(1-|z|^2)^{-1} \log \frac{1}{1-|z|}(1-|\varphi_a(z)|^2)\, dA(z) < \infty\,,$$

then clearly $B^1_{1,\log} \subsetneq B^1_1$.
Moreover, the singular inner function S does not belong to $B^1_{1,\log}$.

As pointed out, the zero-sequence $\{z_n\}$ of a Blaschke product in $B^1_{1,\log}$ satisfies (4.4).
However, complete characterizations of inner functions in B^p_1 and $B^p_{1,0}$ remain as an open problem.