Three Problems for Weighted Bloch-Lipschitz Spaces of Holomorphic Functions on the Unit Ball

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Radial Differential Operators

Let $H(\mathbb{B})$ be the space of holomorphic functions on the unit ball $\mathbb{B}$ of $\mathbb{C}^N$. Every $f \in H(\mathbb{B})$ has a homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$, where f_k is a homogeneous polynomial of degree k.

For $s, t \in \mathbb{R}$, we have invertible radial differential operators D_s^t on $H(\mathbb{B})$ defined by $D_s^t f = \sum_{k=0}^{\infty} d_k f_k$.
Each D_s^t is of order t for any s, $d_k \neq 0$ for all $k = 0, 1, 2, \ldots$, and $d_k \sim k^t$ as $k \to \infty$.
Here $a \sim b$ means $C_1|b| \leq |a| \leq C_2|b|$ for two positive constants.

If p is a polynomial of degree m, then $D_s^t p$ also has degree m.
We also have $D_s^0 = I$ and $(D_s^t)^{-1} = D_{s+t}^{-t}$.

We use $\langle \cdot, \cdot \rangle$ to denote the usual Hermitian inner product on $\mathbb{C}^N$; the corresponding norm is $| \cdot |$. If $z \in \mathbb{C}^N$, then $z = (z_1, \ldots, z_N)$.

The usual radial derivative R is given by $Rf(z) = \langle \nabla f(z), \bar{z} \rangle$.
We have $D_{-N}^1 = I + R$.

The Spaces

Definition 1.

Let $\alpha \in \mathbb{R}$. The *weighted Bloch space* $\mathcal{B}^\alpha$ is defined as the space of those $f \in H(\mathbb{B})$ for which $\|f\|_{B^\alpha} = \sup_{z \in \mathbb{B}} (1-|z|^2)^{\alpha+t} \, |D_s^t f(z)|$ is finite for some s, t satisfying $\alpha + t > 0$.
The *weighted little Bloch space* $\mathcal{B}_0^\alpha$ is that subspace of $\mathcal{B}^\alpha$ whose elements f satisfy $\lim_{|z| \to 1} (1-|z|^2)^{\alpha+t} \, |D_s^t f(z)| = 0$.

The condition $\alpha + t > 0$ ensures that the spaces contain the polynomials and thus are nontrivial. Using Bergman-Besov projections, it can be shown that the spaces are independent of such t and $s > \alpha - 1$, although the norms are only equivalent.

Our use of α is nonstandard and the classical Bloch space is $\mathcal{B}^0$ and the classical little Bloch space is $\mathcal{B}_0^0$ using $t = 1$. Other people write α in place of our $\alpha + t$.

Elementary Relations

For $\alpha < 0$, the spaces $\mathcal{B}^\alpha$ are the holomorphic *Lipschitz spaces* $\Lambda_{-\alpha}$. In particular, $\mathcal{B}^{-1}$ is the *Zygmund class*. See Zhu's book on these connections.

For $\alpha > 0$, $t = 0$ satisfies $\alpha + t > 0$, and no derivatives are needed in the definition of $\mathcal{B}^\alpha$. So $f \in \mathcal{B}^\alpha$ if and only if $(1 - |z|^2)^\alpha \, |f(z)|$ is bounded on $\mathbb{B}$. These spaces are also called the *growth spaces*. In particular $\mathcal{B}^2$ is the *Bers space*.

The definitions immediately imply the following.

Theorem 2.

For any α and s, t, $D_s^t(\mathcal{B}^\alpha) = \mathcal{B}^{\alpha+t}$ is an isomorphism, and also an isometry when norms with appropriate s, t are used in the two spaces.

In this talk we concentrate on results that are valid for all $\alpha \in \mathbb{R}$.

Some Isometries of $\mathcal{B}^\alpha$

Let φ_b be the automorphism of $\mathbb{B}$ (involutive Möbius transformation) that exchanges 0 and $b \in \mathbb{B}$. For $f \in H(\mathbb{B})$ and $\alpha \in \mathbb{R}$, define

$$T_b^\alpha f(z) = \frac{(1-|b|^2)^\alpha}{(1-\langle z, b\rangle)^{2\alpha}} \, f(\varphi_b(z)) = C\, f(\varphi_b(z))\,(J\varphi_b)^{2\alpha/(N+1)},$$

where $|C| = 1$ and J is the complex Jacobian. On $H(\mathbb{B})$, further define

$$W_b^\alpha = D_{s+t}^{-t}\, T_b^{\alpha+t} D_s^t.$$

Replacing φ_b by a general automorphism ψ of $\mathbb{B}$, we have T_ψ^α and W_ψ^α.

Lemma 3.

Given $\alpha \in \mathbb{R}$, pick s, t to satisfy $\alpha + t > 0$, and consider $\mathcal{B}^\alpha$ with the norm using these s, t. Then W_b^α with the same s, t is a linear surjective involutive isometry of $\mathcal{B}^\alpha$.

We use W_b^α to move from $0 \in \mathbb{B}$ to $b \in \mathbb{B}$.

An Extremal Problem

For each $b \in \mathbb{B}$, consider the extremal problem of determining

$$S_\alpha(b) = \sup \{ f(b) > 0 : f \in \mathcal{B}^\alpha, \ \|f\|_{\mathcal{B}^\alpha} = 1 \}$$

and if possible finding a function realizing it. The problem also depends on the s, t used in the norm.

Vukotić solved this problem in weighted Bergman spaces, and Kaptanoğlu extended it to all two-parameter Besov spaces.

Theorem 4.

An extremal function attaining $S_\alpha(b)$ exists and lies in $\mathcal{B}_0^\alpha$. For $b = 0$, it is the constant function 1, and for $b \neq 0$, it is $W_b^\alpha 1 = T_b^{\alpha+t} 1$.

The solution is valid for all $\alpha \in \mathbb{R}$.

α-Möbius Invariance

Definition 5.

Let $(X, \|\cdot\|)$ be a Banach space of holomorphic functions on $\mathbb{B}$. We call X an *α-Möbius-invariant* space if for some s, t satisfying $\alpha + t > 0$, we have $W_\psi^\alpha f \in X$ whenever $f \in X$, $\|W_\psi^\alpha f\| \leq C \|f\|$, and the action $\psi \mapsto W_\psi^\alpha f$ is continuous for $f \in X$ and *unitary* ψ.

Lemma 6.

For $\alpha \neq 0$, an α-Möbius-invariant space X contains all polynomials.

The case $\alpha = 0$ requires also that X contains a nonconstant function; see Zhu's book.

Definition 7.

A nonzero bounded linear functional on X is called *decent* if it extends to be continuous on $H(\mathbb{B})$.

Maximality of $\mathcal{B}^\alpha$

Theorem 8.

The space $\mathcal{B}^\alpha$ contains with continuous inclusion those α-Möbius-invariant spaces that possess a decent linear functional.

Rubel and Timoney proved this for the classical Bloch space ($\alpha = 0$) on the unit disc in $\mathbb{C}^1$, and later Timoney extended it to the case $\alpha = 0$ on bounded symmetric domains. Our result is valid for all $\alpha \in \mathbb{R}$.

Corollary 9.

There is no α-Möbius-invariant closed subspace of $H(\mathbb{B})$ other than $\{0\}$, and also $\mathbb{C}$ for $\alpha = 0$.

Corollary 10.

Suppose $p > 0$ and $q > -1$, or $p \geq 2$ and $-(N+1) \leq q \leq 1$. Suppose also $\alpha > 0$ and $N + 1 + q = \alpha p$. Then the Besov space B_q^p is an α-Möbius-invariant space and $B_q^p \subset \mathcal{B}^\alpha$ with continuous inclusion.

Hermitian Metrics on $\mathbb{B}$

For $-1 < \alpha < 0$, the holomorphic Lipschitz space $B^\alpha = \Lambda_{-\alpha}$ is traditionally defined as the space of those $f \in H(\mathbb{B})$ satisfying $|f(z) - f(w)| \leq C\,|z-w|^{-\alpha}$ for all $z, w \in \mathbb{B}$.

Further, if ρ_0 is the Bergman metric on $\mathbb{B}$, then f belongs to the classical Bloch space B^0 if and only if $|f(z) - f(w)| \leq C\,\rho_0(z,w)$ for all $z, w \in \mathbb{B}$ (Timoney). We do the same for weighted Bloch spaces $\mathcal{B}^\alpha$ for $\alpha > 0$.

For $\alpha > 0$, we define new infinitesimal Hermitian metrics on $\mathbb{B}$ by

$$g_{\alpha_{ij}} = \frac{1}{(1-|z|^2)^{2(1+\alpha)}}\left((1-|z|^2)\,\delta_{ij} + z_i\overline{z}_j\right) \qquad (1 \leq i, j \leq N).$$

We let ρ_α be the corresponding integrated distance function.

It turns out that $\mathbb{B}$ is unbounded in ρ_α for $\alpha > 0$, that is, geodesics can be continued indefinitely, that is, $(\mathbb{B}, \rho_\alpha)$ is geodesically complete. Then by the Hopf-Rinow theorem, $(\mathbb{B}, \rho_\alpha)$ is a complete metric space, its closed and bounded subsets are compact, and there exists a geodesic of g_α joining any two points of $\mathbb{B}$.

Properties of the Metrics

The *Laplace-Beltrami operator* associated to g_α is

$$\tilde{\Delta}_\alpha = \sum_{i,j=1}^{N} g_\alpha^{ij} \frac{\partial^2}{\partial z_i \, \partial \overline{z}_j} + 4\,(N-1)\,\alpha\,(1-|z|^2)^{1+2\alpha}\,(R+\overline{R}),$$

where $[g_\alpha^{ij}]$ is the inverse if $[g_{\alpha_{ij}}]$. The presence of the first-order terms in $\tilde{\Delta}_\alpha$ for $\alpha > 0$ and $N = 1$ shows that g_α is not a *Kähler metric*. This means that it can*not* be obtained by differentiation as

$$g_{0_{ij}}(z) = \frac{1}{N+1} \frac{\partial^2 \log K(z,z)}{\partial \overline{z}_i \, \partial z_j}$$

unless $\alpha = 0$ or $N = 1$; here K is the Bergman kernel for $\mathbb{B}$.

But for $N = 1$ and, say, $\alpha = 1$, a Kähler potential for g_1 is

$$L_1(z,w) = \frac{1}{3}\left(\log \frac{1}{1-z\overline{w}} + \frac{1}{1-z\overline{w}} + \frac{1}{2(1-z\overline{w})^2}\right).$$

Lipschitz Property for $\alpha > 0$

The *(holomorphic sectional) curvature* of g_α at $z \in \mathbb{B}$ for $\alpha > 0$ is negative, but tends to 0 as $|z| \to 1$. For $N = 1$, the curvature is

$$\kappa_\alpha(z) = -\frac{\Delta \log g_\alpha(z)}{g_\alpha(z)} = -4\,(1+\alpha)\,(1-|z|^2)^{2\alpha}.$$

The explicit form of ρ_α can be obtained when $N = 1$; for $\alpha = 1$, it is

$$\rho_1(z,w) = \frac{1}{4}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} + \frac{1}{2}\frac{|\varphi_z(w)|\,(1-2\,\mathrm{Re}(\bar{z}\,\varphi_z(w))+|z|^2)}{1-|\varphi_z(w)|^2}.$$

Our final result is the following Lipschitz property.

Theorem 11.

For $\alpha > 0$, if $f \in \mathcal{B}^\alpha$, then $|f(z)-f(w)| \le C\,\rho_\alpha(z,w)$ for all $z, w \in \mathbb{B}$. The converse also holds for $N = 1$.