Schwarzian derivative for harmonic mappings

Rodrigo Hernández

Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez
Viña del Mar, Chile
Joint work with María José Martín (UAM)

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Part I

Analytic case

Definition

Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a locally univalent analytic mapping ($f' \neq 0$) and Ω simply connected domian, the Schwarzian derivative is defined as

$$Sf = (f''/f')' - (1/2)(f''/f')^2 .$$

This operator characterizes the Möbius transformations: $Sf \equiv 0$ if and only if $f = T$, where T is a Möbius transformation given by

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

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Properties

If $f = h \circ g$ it follows that

$$Sf = (Sh \circ g)(g')^2 + Sg\,.$$

This implies that $Sf(a)(1-|a|^2)^2 = S(f \circ \phi_a)(0)$, where $\phi_a(z) = \dfrac{a-z}{1-\overline{a}z}$.

Then, the norm of the schwarzian derivative of $f : \mathbb{D} \to \mathbb{C}$ is define by:

$$\|Sf\| = \sup_{z \in \mathbb{D}} (1-|z|^2)^2 |Sf(z)|$$

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Theorem (Nehari)

Let $f : \mathbb{D} \to \mathbb{C}$ be a locally univalent mapping. If $\|Sf\| \leq 2$ then f is univalent in $\mathbb{D}$.

Theorem (Krauss)

Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent mapping, then If $\|Sf\| \leq 6$.

Moreover, if $f(\mathbb{D})$ is a convex domain, then $\|Sf\| \leq 2$.

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Prescribing the Schwarzian derivative

Given $Sf = 2p$, then $f = u/v$ such that u and v are linearly independent solutions of

$$u'' + pu = 0\,.$$

Lemma: f is univalent iff the equation $u'' + pu = 0$ is disconjugate.

Remark: This is one of the way to prove the Nehari's theorem.

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Best Möbius approximation

(Tamanoi, Math Ann. 1996)
Given f we want to find a möbius T such that $ad - bc \neq 0 (= 1)$ and $f(0) = T(0)$, $f'(0) = T'(0)$, $f''(0) = T''(0)$.

This produce a function $F = T^{-1} \circ f$ which satisfies that

$$F(z) = (z - w) + Sf(w)(z - w)^3 + \cdots + S_n(w)(z - w)^n \,,$$

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Part II

Harmonic case

Harmonic mappings

A complex-value harmonic mapping is define as $f = u + iv$ where $\triangle u = \triangle v = 0$.

In a simply connected domain f has the representation $f = h + \overline{g}$, where h and g are analytic. This representation is unique up to an additive constant.

f is locally univalent and sense preserving wherever $J_f = |h'|^2 - |g'|^2 > 0$ (this implies that $h' \neq 0$).

The complex dilatation of f is given by $\omega = g'/h'$. In this talk we consider $|\omega| < 1$ (sense preserving mappings).

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The complex dilatation of f is given by $\omega = g'/h'$. In this talk we consider $|\omega| < 1$ (sense preserving mappings).

Formula and Properties

The möbius harmonic mappings are define as:

$$M = T + \alpha \overline{T}\,,$$

where $T(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0\,.$

Using the Tamanoi's ideas in the harmonic case, in order to find a M such that $T(0) = h(0)$, $T'(0) = h'(0)$, $M_{\overline{z}}(0) = \alpha \overline{T'(0)} = \overline{g'(0)}$, $T''(0) = h''(0)$, we obtain that the analogous of the schwarzian derivative is given by:

Definition

Let $f = h + \overline{g}$ be a sense preserving harmonic mapping with complex dilatation ω, we define the Schwarzian derivative as:

$$S_f = Sh + \frac{\overline{\omega}}{1-|\omega|^2}\left(\omega' \frac{h''}{h'} - \omega''\right) - \frac{3}{2}\left(\frac{\omega' \overline{\omega}}{1-|\omega|^2}\right)^2 .$$

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Properties

- $S_f \equiv 0$ iff f is a Möbius harmonic mapping.
- S_f is analytic iff $f = h + \alpha \overline{h}$, where $|\alpha| < 1$.
- Let φ be analytic function, then $S_{f\circ\varphi}(z) = S_f(\varphi(z))(\varphi'(z))^2 + S\varphi(z)$.
- Let $L(z) = az + b\overline{z} + c$ an affine mapping with $a \neq 0$ and $|b/a| < 1$ (sense preserving), then $S_{L\circ f} = S_f$.
- If S_f is harmonic, then S_f is analytic.

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Results

Theorem

Let f and F be a sense preserving harmonic mappings with complex dilatations ω_f and ω_F respectivelly. Then

- $S_f = S_F$ iff $J_f = cJ_F$, for some constant c.
- $S_f = S_F$ iff $\dfrac{|\omega_f'|}{(1-|\omega_f|^2)} = |c|\dfrac{|\omega_F'|}{(1-|\omega_F|^2)}$, for some constant c.

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Let $f = h + \overline{g}$ be a sense preserving harmonic mapping define in $\mathbb{D}$, then $\|S_f\| < \infty$ iff $\|Sh\| < \infty$.

Convex mappings

Let $f = h + \overline{g}$ be a sense preserving convex harmonic mapping ($f(\mathbb{D})$ is convex), then $\|S_f\| < 6$.

Theorem

There exists a constant C such that $\|S_f\| < C$ for all sense preserving univalent harmonic mappings.

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Some open problems

- Given S_f, can we recovered the function f?
- There exists a constant C such that $\|S_f\| < C$ implies that f is univalent ?
- Find a sharp bound of the $\|S_f\|$ for all univalent mappings. We conjectured that the bound is $19/2$.

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