

Besov spaces, multipliers, and univalent functions

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$f \in Hol(\mathbb{D})$ belongs to the Besov space B^p , $1 < p < \infty$, if

$$\|f\|_p = |f(0)| + \left((p-1) \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) \right)^{1/p} < \infty.$$

The Bloch space \mathcal{B} is the set of analytic functions f such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| (1-|z|^2) < \infty.$$

The analytic function $f \in \Lambda_1^1$ if

$$\|f\|_{\Lambda_1^1} = |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} |f'(e^{i\theta})| d\theta < \infty.$$

The minimal space B^1

For $f \in Hol(\mathbb{D})$, we have that $f \in B^1$ if $\exists \{a_k\}_{k=1}^{\infty} \in \mathbb{D}$ and $\{\lambda_k\}_{k=1}^{\infty} \in \ell^1$ such that

$$f(z) = \lambda_0 + \sum_{k=1}^{\infty} \lambda_k \varphi_{a_k}(z), \quad z \in \mathbb{D}, \quad (1)$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Furthermore,

$$\begin{aligned} \|f\|_1 &= \inf \left\{ \sum |\lambda_k| : (1) \text{ holds} \right\} \\ &\asymp |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)| dA(z). \end{aligned}$$

Multiplication operators

For $g \in Hol(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f) = g(z)f(z), \quad f \in Hol(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let X and Y be two Banach spaces of analytic functions in \mathbb{D} that contain the polynomials and are continuously embedded in $Hol(\mathbb{D})$.

Pointwise multipliers

A function $g \in Hol(\mathbb{D})$ is called a pointwise multiplier from X into Y if $M_g(X) \subset Y$. The space of all such multipliers will be denoted by $M(X, Y)$ or $M(X)$ when $X = Y$.

Known results

- $M(B^1) = B^1$.
- $M(\Lambda_1^1) = \Lambda_1^1$.
- $g \in M(\mathcal{B}) = \mathcal{B}$ if and only if g is bounded in \mathbb{D} and

$$\sup_{z \in \mathbb{D}} |g'(z)| (1 - |z|^2) \log \frac{2}{1 - |z|^2} < \infty.$$

- Characterization of $M(\mathcal{B}^P)$ in terms of certain capacities.

Galanopoulos, Girela, and Peláez

$$M(B^p, B^q) = \{0\}, \quad \text{if } 1 \leq q < p.$$

Axler and Shields

$$g \in M(\mathcal{D}) \Rightarrow \sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right)^{1/2} < \infty.$$

Zorboska

$$g \in M(\mathcal{B}^p), 1 < p < \infty \Rightarrow$$

$$\sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right)^{1 - \frac{1}{p}} < \infty.$$

Zorboska

Suppose that $1 < p < \infty$ and $g \in H^\infty$.

- (i) If $g \in M(B^p)$ and $0 < r < 1$, then

$$\sup_{w \in \mathbb{D}} \int_{\Delta(w,r)} |g'(z)|^p (1 - |z|^2)^{p-2} \left(\log \frac{2}{1 - |z|^2} \right)^{p-1} dA(z)$$

is finite.

- (ii) If

$$\int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p-2} \left(\log \frac{2}{1 - |z|^2} \right)^{p-1} dA(z) < \infty,$$

then $g \in M(B^p)$.

Zorboska

Suppose that $1 < p < \infty$ and $g \in H^\infty$.

(i) If $g \in M(B^p)$ and $0 < r < 1$, then

$$\sup_{w \in \mathbb{D}} \int_{\Delta(w,r)} |g'(z)|^p (1 - |z|^2)^{p-2} \left(\log \frac{2}{1 - |z|^2} \right)^{p-1} dA(z)$$

is finite. This condition is equivalent to say that the measure μ_g defined by
 $d\mu_g(z) = |g'(z)|^p (1 - |z|^2)^{p-2} dA(z)$ satisfies

$$\mu_g(S(I)) \leq C \left(\log \frac{2}{|I|} \right)^{1-p},$$

where I is an interval in $\partial\mathbb{D}$ and $S(I)$ its associated Carleson box.



Theorem

- (i) $M(B^1, \mathcal{B}) = H^\infty$.
- (ii) $M(B^1, B^q) = B^q \cap H^\infty, \quad 1 < q < \infty$.
- (iii) If $1 < p < \infty$ then $g \in M(B^p, \mathcal{B})$ if and only $g \in H^\infty$ and

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| \left(\log \frac{2}{1 - |z|^2} \right)^{1 - \frac{1}{p}} < \infty.$$

- (iv) If $1 < p < q < \infty$ then $g \in M(B^p, B^q) \iff$ if $g \in H^\infty$ and for each $r \in (0, 1)$

$$\sup_{w \in \mathbb{D}} \int_{\Delta(w, r)} (1 - |z|^2)^{q-2} |g'(z)|^q \left(\log \frac{2}{1 - |z|^2} \right)^{q(1 - \frac{1}{p})} dA(z)$$

is finite.



Theorem

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- (iii) If $1 < p < \infty$ then $g \in M(B^p, \mathcal{B})$ if and only $g \in H^\infty$ and

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| \left(\log \frac{2}{1 - |z|^2} \right)^{1 - \frac{1}{p}} < \infty.$$

- (iv) If $1 < p < q < \infty$ then $g \in M(B^p, B^q) \iff g \in H^\infty$ and for each $r \in (0, 1)$

$$\sup_{w \in \mathbb{D}} \int_{\Delta(w, r)} (1 - |z|^2)^{q-2} |g'(z)|^q \left(\log \frac{2}{1 - |z|^2} \right)^{q(1 - \frac{1}{p})} dA(z)$$

is finite.



Theorem

- (i) $M(B^1, \Lambda_1^1) = M(\Lambda_1^1) = \Lambda_1^1.$
- (ii) $M(\Lambda_1^1, B^q) = B^q \cap H^\infty,$ for all $q > 1.$
- (iii) $M(B^p, \Lambda_1^1) = \{0\},$ for all $p > 1.$
- (iv) $M(\Lambda_1^1, B^1) = \{0\}.$

Theorem

- (i) $B^1 \notin M(B^p, B^q)$, $1 < p \leq q < \infty$.
- (ii) $B^1 \notin M(B^p, \mathcal{B})$, $1 < p < \infty$.
- (iii) $B^1 \notin M(\mathcal{B})$.

Corollary

The spaces $M(B^p, B^q)$ ($1 < p \leq q < \infty$), $M(B^p, \mathcal{B})$, ($1 < p < \infty$), and $M(\mathcal{B})$ are not conformally invariant.

For $\beta > 0$ and $k = 1, 2, \dots$, we set

$$a_k = 1 - e^{-e^k}, \quad \lambda_{k,\beta} = \left(\log \frac{1}{1 - |a_k|} \right)^{-\beta} = e^{-k\beta}.$$

We let f_β be the analytic function in \mathbb{D} defined by

$$f_\beta(z) = \sum_{k=1}^{\infty} \lambda_{k,\beta} \varphi a_k(z), \quad z \in \mathbb{D}.$$

- If $1 < p < \infty$ and $0 < \beta < 1 - \frac{1}{p}$, then $f_\beta \notin M(B^p, B^q)$ ($p \leq q < \infty$) and $f_\beta \notin M(B^p, \mathcal{B})$.
- If $0 < \beta < 1$, then $f_\beta \notin M(\mathcal{B})$.

Definition

By a **lacunary series** we mean a power series of the form

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \quad z \in \mathbb{D}$$

with $n_{k+1} \geq \lambda n_k$ (for a certain $\lambda > 1$).

We use \mathcal{L} to denote the class of functions $f \in Hol(\mathbb{D})$ given by a lacunary power series.

$$g \in \mathcal{L} \cap B^p \Leftrightarrow \sum_{k=0}^{\infty} n_k |a_k|^p < \infty.$$

Theorem

Suppose that $1 \leq p \leq q < \infty$ and $g(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathcal{L}$. Then, the following conditions are equivalent:

- (i) $g \in B^q$.
- (ii) $g \in M(B^p, B^q)$.
- (iii) $\sum_{k=0}^{\infty} n_k |a_k|^q < \infty$.

In particular, we have

$$\mathcal{L} \cap B^p = \mathcal{L} \cap M(B^p), \quad 1 \leq p < \infty.$$

We use \mathcal{U} to denote the set of univalent functions in \mathbb{D} (analytic and one-to-one functions in the unit disk).

Axler and Shields gave an example of a function $g \in (\mathcal{U} \cap \Lambda_1^1) \setminus M(\mathcal{D})$. Thus,

$$\mathcal{U} \cap \Lambda_1^1 \not\subset M(\mathcal{D}).$$

Theorem

If $1 < p \leq q < \infty$, then $\mathcal{U} \cap \Lambda_1^1 \not\subset M(B^p, B^q)$.

Axler and Shields

If g is a univalent mapping onto a bounded starlike domain, then $g \in M(\mathcal{D})$.

Theorem

Let g be a univalent mapping onto a bounded starlike domain Ω .

- (i) If $1 \leq p \leq q$ and $q \geq 2$, then $g \in M(B^p, B^q)$. In particular, $g \in M(B^q)$.
- (ii) If $1 \leq p \leq q < 2$, there exists $g \in \mathcal{U}$ onto a starlike domain that does not belong to $M(B^p, B^q)$.
- (iii) If Ω turns out to be convex, then $g \in M(B^p, B^q)$ whenever $1 \leq p \leq q$.