SCHWARZIAN DERIVATIVE AND GENERAL BESOV-TYPE DOMAINS

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$H(\mathbb{D})$ is the space of functions analytic on the open unit disk $\mathbb{D}$;

$f \in H(\mathbb{D})$ is *univalent* when it is one-to-one on $\mathbb{D}$;

$\mathcal{B} = \{f \in H(\mathbb{D}) : ||f||_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty\}$ is the *Bloch space*;

• For g in $H(\mathbb{D})$, $g \in \mathcal{B}$ iff $\exists c \in \mathbb{C}$ and a univalent f such that $g = c \log f'$. Then

$$||g||_{\mathcal{B}} = |c| \sup_{z\in\mathbb{D}}(1 - |z|^2)|\frac{f''(z)}{f'(z)}| \leq 6|c|.$$

What can we say about a univalent f, the domain $\Omega = f(\mathbb{D})$, or the boundary curve $\partial\Omega$, given that $\log f' \in X$, for some Banach space $X \subseteq \mathcal{B}$?

For f univalent, we say that $\Omega = f(\mathbb{D})$ is an X-*domain* whenever $\log f' \in X$.

For example, if Ω is the inner domain of a (closed) Jordan curve $\partial\Omega$:

• Ω is a $\mathcal{B}_0$ domain iff $\partial\Omega$ is asymptotically conformal (Pommerenke, '78);

• Ω is a $VMOA$ domain iff $\partial\Omega$ is asymptotically smooth (Pommerenke, '78);

• If $\partial\Omega$ is rectifiable and $||\log f'||_{\mathcal{B}}$ is small enough, then Ω is a $BMOA$ domain iff $\partial\Omega$ is a quasi-smooth (Lavrentiev) curve (Pommerenke, '77).

Other characterizations given in terms of the Schwarzian derivative S_f:

• Ω is a $\mathcal{B}_0$ domain iff $(1-|z|^2)^2 S_f(z) \to 0$, as $|z| \to 1^-$ (Pommerenke, '78);

• Ω is a $BMOA$ domain iff $(1-|z|^2)^3 |S_f(z)|^2 dA(z)$ is a Carleson measure on $\mathbb{D}$ (Astala, Zinsmeister, '91).

Let f be (locally) univalent function in $H(\mathbb{D})$. Then

$P_f(z) = (\log f')'(z) = \frac{f''(z)}{f'(z)}$ (*pre-Schwarzian derivative*),

$S_f(z) = P_f'(z) - \frac{1}{2}(P_f(z))^2 = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$ (*Schwarzian derivative*).

Properties of P_f and S_f:

• If f is univalent on $\mathbb{D}$, then

$$(1-|z|^2)|P_f(z)| \leq 6 \text{ and } (1-|z|^2)^2|S_f(z)| \leq 6.$$

• If $(1-|z|^2)|zP_f(z)| \leq 1$, or $(1-|z|^2)^2|S_f(z)| \leq 2$, then f is univalent.

• The Schwarzian derivative is Möbius invariant , i.e. $S_{\psi_a \circ f} = S_f$, and

$$(1-|z|^2)^2|S_{f\circ\psi_a}(z)| = (1-|\psi_a(z)|^2)^2|S_f(\psi_a(z))|,$$

for every Möbius transformation $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$.

Recall:

For $s > 0$, a positive measure μ on $\mathbb{D}$ is an $s-$ *Carleson measure* whenever

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\psi_a'(z)|^s \, d\mu(z) < \infty,$$

or equivalently, whenever

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty,$$

where I is a subarc of the unit circle with normalized length $|I|$, and $S(I)$ is a Carleson box determined by I, i.e.

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I\}.$$

For $s > 0$, a positive measure μ on $\mathbb{D}$ is a *vanishing* $s-$ *Carleson measure* whenever

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |\psi_a'(z)|^s \, d\mu(z) = 0,$$

or equivalently, whenever

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0.$$

For $p > 1$ and $s \geq 0$, the analytic Besov-type space $B_{p,s}$ is defined by

$$B_{p,s} = \{f \in H(\mathbb{D}) : \|f\|^p_{B_{p,s}} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p (1-|\psi_a(z)|^2)^s d\lambda(z) < \infty\},$$

where

$$\psi_a(z) = \frac{a-z}{1-\bar{a}z}$$

is a Möbius transformation related to the point $a \in \mathbb{D}$, and $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is a Möbius invariant measure on $\mathbb{D}$.

$B_{p,s}$ is a Banach space with a Möbius invariant seminorm $\|\cdot\|_{B_{p,s}}$. As shown by Rubel and Timoney, every Möbius invariant function space must be included in the Bloch space $\mathcal{B}$.

The Besov-type spaces $B_{p,s}$ are contained in a more general class of so called $F_{p,q,s}$ spaces, introduced by R. Zhao.

For $p > 0$, $s \geq 0$ and $q > -2$,

$$\|f\|^p_{F_{p,q,s}} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\psi_a(z)|^2)^s dA(z)$$

and

$$F_{p,q,s} = \{f \in H(\mathbb{D}) : \|f\|^p_{F_{p,q,s}} < \infty\}.$$

For $s > 0$, let

$$F^0_{p,q,s} = \{f \in H(\mathbb{D}) : \lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\psi_a(z)|^2)^s dA(z) = 0\}.$$

and for $s = 0$, define $F^0_{p,q,0} = F_{p,q,0}$.

• If $q + s > -1$ and $p \geq 1$, then $F_{p,q,s}$ is a nontrivial Banach space contained in the Bloch-type space $\mathcal{B}^\alpha$ with $\alpha = \frac{q+2}{p}$, where

$$\mathcal{B}^\alpha = \{f \in H(\mathbb{D}), ||f||_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty\}.$$

Moreover, $F^0_{p,q,s} \subseteq \mathcal{B}_0^{\frac{q+2}{p}}$, where for $\alpha > 0$

$$\mathcal{B}^\alpha_0 = \{f \in H(\mathbb{D}) : \lim_{|z|\to 1} (1-|z|^2)^\alpha |f'(z)| = 0\}.$$

• If $s > 1$, then $F_{p,q,s} = \mathcal{B}^{\frac{q+2}{p}}$, $F^0_{p,q,s} = \mathcal{B}_0^{\frac{q+2}{p}}$ and so for $s > 1, q + 2 = p$, we have that $F_{p,q,s} = \mathcal{B}$ and $F^0_{p,q,s} = \mathcal{B}_0$.

Hence, we will always assume that $q + s > -1$, $1 \leq p < \infty$ and $q + 2 \leq p$ so that $F_{p,q,s} \subset \mathcal{B}$, and will refer to this range of the parameters as the *standard range* of p, q and s.

• For $p > 1$, $F_{p,p-2,s} = B_{p,s}$, and so $F_{p,p-2,0} = B_p$, the analytic Besov space.

• When $p = 2$, $F_{2,0,s} = Q_s$. Thus, $F_{2,0,1} = BMOA$ and $F^0_{2,0,1} = VMOA$.

Characterizations of X-domains in terms of Carleson measure conditions involving the Schwarzian derivative:

- (Astala, Zinsmeister, '91) $BMOA$ domains.
- (Pau, Peláez '09) Q_s domains.
- (Pérez-González, Rättyä, '09) B_p domains and $Q_{s,0}$ domains.
- (Galanopolous, Girela, Hernández, '11) B_p (general) domains.

We have the following characterization of $F_{p,q,s}$ domains:

Theorem 1. Let f be univalent on $\mathbb{D}$, $1 \leq p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$ and $q + s > -1$. Let $\Omega = f(\mathbb{D})$, and let $\partial\Omega$ be a Jordan curve.
If $q + 2 = p$, then Ω is a $F_{p,q,s}$ domain if and only if

$$d\mu_{f,p,q,s}(z) = |S_f(z)|^p(1 - |z|^2)^{p+q+s}dA(z)$$

is an s-Carleson measure.
If $q+2 < p$, then Ω is a $F_{p,q,s}$ domain if and only if $\log f' \in \mathcal{B}_0$ and $d\mu_{f,p,q,s}(z)$ is an s-Carleson measure.

Theorem 2. Let f be univalent on $\mathbb{D}$, $1 \leq p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and $q + s > -1$. Let $\Omega = f(\mathbb{D})$, and let $\partial\Omega$ be a Jordan curve.
If $q + 2 \leq p$, then Ω is a $F^0_{p,q,s}$ domain if and only if

$$d\mu_{f,p,q,s}(z) = |S_f(z)|^p(1 - |z|^2)^{p+q+s}dA(z)$$

is a vanishing s-Carleson measure.

A bit on the techniques and ideas of the proof:

Lemma 1. **[Rättyä, '03]** Let p, q and s be constants in the standard range, let $n \in \mathbb{N}$, or $n = 0$ and $q + s - p > -1$, and let $h \in H(\mathbb{D})$. Then the following are equivalent:

(i) h is a function in $F_{p,q,s}$.

(ii) $|h^{(n)}(z)|^p(1-|z|^2)^{(n-1)p+q+s}dA(z)$ is an s-Carleson measure.

(iii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h^{(n)}(z)|^p(1-|z|^2)^{(n-1)p+q+s}|\psi_a'(z)|^s dA(z) < \infty$.

Lemma 2. Let p, q and s be constants in the standard range.
If $\log f' \in F_{p,q,s}$, then

$$d\mu_{f,p,q,s}(z) = |S_f(z)|^p(1-|z|^2)^{p+q+s}dA(z)$$

is an s-Carleson measure.

Proof. Recall:
$P_f(z) = (\log f')'(z)$; $S_f(z) = P_f'(z) - \frac{1}{2}(P_f(z))^2$; $||\log f'||_{\mathcal{B}} \le 6$.

By Lemma 1, $\log f' \in F_{p,q,s}$ iff $|P_f(z)|^p(1-|z|^2)^{q+s}dA(z)$ is s-Carleson measure iff $|P_f'(z)|^p(1-|z|^2)^{p+q+s}dA(z)$ is s-Carleson measure.
Moreover, for $p \ge 1$,

$$|S_f(z)|^p(1-|z|^2)^{p+q+s} \le$$
$$2^{p-1}|P_f'(z)|^p(1-|z|^2)^{p+q+s} + \frac{1}{2}||\log f'||_{\mathcal{B}}^p|P_f(z)|^p(1-|z|^2)^{q+s}.$$

$\square$

Lemma 3. Let p, q and s be constants in the standard range. Let f be such that $\Omega = f(\mathbb{D})$ is a Jordan domain, and such that $\log f' \in \mathcal{B}_0$. Then if

$$d\mu_{f,p,q,s}(z) = |S_f(z)|^p (1-|z|^2)^{p+q+s} dA(z)$$

is an s-Carleson measure, we have that $|P_f(z)|^p(1-|z|^2)^{q+s}dA(z)$ is also an s-Carleson measure.

Proof. Since $\Omega = f(\mathbb{D})$ is a Jordan domain and $\log f'$ is in $\mathcal{B}_0$, we have that $\partial\Omega$ is asymptotically conformal. Hence, $\forall \varepsilon > 0$, $\exists r_\varepsilon > 0$ such that whenever $|z| > r_\varepsilon$, we have that $(1-|z|^2)|P_f(z)| < \varepsilon$.

Using small enough ε, the proof follows by showing that $\exists C_1(\varepsilon), C_2(\varepsilon) > 0$ such that

$$\int_{\mathbb{D}} |P_f'(z)|^p (1-|z|^2)^{p+q+s} |\psi_a'(z)|^s dA(z) \leq$$
$$C_1(\varepsilon) \int_{\mathbb{D}} |S_f(z)|^p (1-|z|^2)^{p+q+s} |\psi_a'(z)|^s dA(z) + C_2(\varepsilon).$$

□

The proof of Theorem 1. for the range of the parameters p, q, s where either $q+2 < p$, or where $q+2 = p$ and $s = 0$, thus covering the Bloch-type spaces $\mathcal{B}^\alpha, 0 < \alpha < 1$ and the Besov spaces $B_p, p > 1$ follows directly from the given lemmas, since these spaces are all included in the little Bloch space $\mathcal{B}_0$.

The remaining range $q + 2 = p$ and $0 < s \leq 1$, covers the case of the Möbius invariant Besov-type spaces $B_{p,s}$.

The corresponding part of the proof of Theorem 1. uses the decomposition of the unit disk into dyadic Carleson squares, and on further estimates and comparisons of the pre-Schwarzian and the Schwarzian derivatives behaviour over the special Carlesons squares.

These techniques have been used before in the characterizations of X-domains in terms of Carleson measure conditions involving the Schwarzian derivative, such as for example in the Astala, Zinsmeister characterization of BMOA domains, or in the proof of the recent result of Pau and Peláez in the characterization of the Q_s domains.

One of the interesting questions, that has been considered in a number of cases is when is the Jordan curve $\partial\Omega$ rectifiable.

Note that even when $\partial\Omega$ is asymptotically conformal, it still does not have to be rectifiable.

- If $\Omega = f(\mathbb{D})$ is a quasidisk, then $\partial\Omega$ is rectifiable iff $\int_{\mathbb{D}} |f'(z)||S_f(z)|^2(1-|z|^2)^3 dA(z) < \infty$ (Bishop-Jones, '94).

- If $\log f'$ is in the space $F^0_{2,0,1} = VMOA$, then $\partial\Omega$ is rectifiable (Pommerenke, '78).

Since the Besov spaces $B_p, 1 \leq p < \infty$ and the spaces $Q_{s,0}, 0 < s \leq 1$ are all contained in $VMOA$, if $\log f'$ is in any of these spaces, then $\partial\Omega$ must also be rectifiable.

We have the following result related to rectifiability of the boundary Jordan curve, which includes the cases mentioned above.

Theorem 3. Let f be univalent on $\mathbb{D}$, let $\Omega = f(\mathbb{D})$ and let $\partial\Omega$ be a Jordan curve.Let p, q and s be in the standard range and let either $0 \leq s < 1$, or $s = 1$ and $q + 2 < p$. If $\log f' \in F^0_{p,q,s}$, then $f' \in H^r$ for all $r > 0$, which further more implies that the Jordan curve $\partial\Omega$ is rectifiable.

Question: If $s = 1$, $q + 2 = p$, $p > 2$, $\log f' \in F^0_{p,q,s}$ and $\Omega = f(\mathbb{D})$ is a Jordan domain, is $\partial\Omega$ rectifiable?

Recall that if, for example, $\log f'$ belongs to $\mathcal{B}_0$, then $\partial\Omega$ is asymptotically conformal and so, f has a quasiconformal extension to the complex plane.

The case when f has a quasiconformal extension is of particular interest in Teichmüller theory. For example, if $S = \{\log f' : f \text{ univalent on } \mathbb{D}\}$ and

$$T(1) = \{\log f' : f \text{ has quasiconformal extension to } \mathbb{C}\},$$

then $T(1)$ is the interior of S in the Bloch norm.

If $S_{BMOA} = \{\log f' : f \text{ univalent on } \mathbb{D}, \log f' \in BMOA\}$, then the interior of S_{BMOA} in the BMOA norm is $S_{BMOA} \cap T(1)$. Also, there is a descriptions of the connected components of $S_{BMOA} \cap T(1)$ (Astala, Zinsmeister, '91).

We have given Schwarzian derivative characterizations of the spaces

$$S_X = \{\log f' : f \text{ univalent on } \mathbb{D}, \log f' \in X\},$$

where X is either an $F_{p,q,s}$, or $F^0_{p,q,s}$ space, contained in the Bloch space.

If further more $F_{p,q,s} \subset \mathcal{B}_0$, or $F^0_{p,q,s} \in \mathcal{B}_0$, we have that $S_X \cap T(1) = S_X$.

Thus, the main interest are the leftover options, i.e. the cases when X is one of the spaces $B_{p,s}, 1 \leq p < \infty, 0 < s \leq 1$.There are many interesting questions related to the topological structure of these types of general Teichmüller spaces.

- Is it always true that $S_{B_{p,s}} \cap T(1)$ is the interior of $S_{B_{p,s}}$ in the $B_{p,s}$ norm, and what is their closure in the $B_{p,s}$ norm, or in the Bloch norm?
- Are there specific descriptions of some of the connected components of $S_{B_{p,s}} \cap T(1)$, in terms of special conditions imposed on f?

The restriction on the parameters p, q, s in the standard range excludes the minimal Möbius invariant space B_1, defined by

$$B_1 = \{f \in H(\mathbb{D}) : \|f\|_{B_1} = \int_{\mathbb{D}} |f''(z)| dA(z) < \infty\}.$$

Note that in the spirit of Lemma 1, the space B_1 can be regarded as on of the $F_{p,q,s}$ spaces, where we take $n = 2$, $p = 1$, $s = 0$ and $q = -1$.

Similar characterization of B_1 domains still holds. One direction of the result, for not necessarily Jordan domains, appears also in (Galanopolous, Girela, Hernández, '11).

Proposition 1. For f univalent on $\mathbb{D}$ and $\Omega = f(\mathbb{D})$ a Jordan domain, $\log f'$ is in B_1 if and only if S_f belongs to the Bergman space A^1, where

$$A^1 = \{h \in H(\mathbb{D}) : \|h\|_{A^1} = \int_{\mathbb{D}} |h(z)| dA(z) < \infty\}.$$

Proof. If S_f is in A^1, i.e $|S_f(z)|dA(z)$ is a finite measure, then $|S_f(z)|(1-|z|^2)^2 \to 0$ as $|z| \to 1$, and so $\log f' \in \mathcal{B}_0$. Similarly to the proof of Lemma 3, this time with $s = 0$, we get that if S_f is in A^1, then P_f' is also in A_1. But $P_f' = (\log f')''$, and so $\log f'$ is in B_1.

For the converse, if $\log f'$ is in B_1, then $\log f'$ is also in the Dirichlet space B_2, since $B_1 \subset B_2$. This implies that $(\log f')' = P_f$ is in the Bergman space A^2, which is equivalent to P_f^2 belonging to A^1. Since $P_f' = (\log f')''$ belongs to A^1, and $S_f = P_f' - \frac{1}{2}P_f^2$, we get that S_f also belongs to A^1. $\square$