Strong continuity of composition semigroups in spaces of analytic functions

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Introduction

$\mathbb{D}$ = unit disc in $\mathbb{C}$.

Def. A semigroup is a family $\{\varphi_t\}_{t\geq 0}$ of analytic self-maps of $\mathbb{D}$ such that.
(i) $\varphi_0 \equiv z$, (identity on $\mathbb{D}$),
(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
(iii) $\varphi_t \to \varphi_0$ as $t \to 0$, uniformly on compact subsets of $\mathbb{D}$.

Generator of $\{\varphi_t\}$:

$$G(z) = \lim_{t\to 0} \frac{\varphi_t(z) - z}{t} = \frac{\partial \varphi_t(z)}{\partial t}\Big|_{t=0}$$

$$= (\overline{b}z - 1)(z - b)F(z),$$

$b \in \overline{\mathbb{D}}$, Denjoy-Wolff point of $\{\varphi_t\}$,

$\operatorname{Re} F(z) \geq 0$.

For such $\{\varphi_t\}$ let

$$C_t(f) = f \circ \varphi_t, \qquad f \text{ analytic on } \mathbb{D}.$$

$(X, \| \|_X)$ a Banach space of analytic functions. Assume

$$C_t : X \to X$$

are bounded operators for $t \geq 0$.

Strong continuity.

When is $\{C_t\}$ strongly continuous (s. c.) on X ?

Need

$$\lim_{t\to 0} \|C_t(f) - f\|_X = 0, \quad \text{each } f \in X.$$

To get s.c. we have to pass from

$$(f \circ \varphi_t)(z) \to f(z), \qquad z \in \mathbb{D},$$

to convergence in $\| \ \|_X$.

1. The easy cases.

Suppose polynomials are dense in X, and

$$\sup_{0<t<\delta} \|C_t\|_{X\to X} = M < \infty, \quad \text{some } \delta > 0,$$

then for a polynomial P and small t,

$$\begin{aligned} \|f\circ\varphi_t - f\| &\le \|f\circ\varphi_t - P\circ\varphi_t\| + \|P\circ\varphi_t - P\| + \|P - f\| \\ &\le (\|C_t\|+1)\|P-f\| + \|P\circ\varphi_t - P\| \end{aligned}$$

Thus

$$\begin{aligned} \text{s. c.} &\Leftrightarrow \lim_{t\to 0} \|P\circ\varphi_t - P\| = 0, \\ &\Leftrightarrow \lim_{t\to 0} \|\varphi_t(z)^k - z^k\| = 0, \quad k = 1, 2, \cdots \end{aligned}$$

Suppose also each $m \in H^\infty$ acts as bounded multipliers of X of norm $\le \|m\|_\infty$. Then

$$\varphi_t(z)^k - z^k = (\varphi_t(z) - z)b_t(z),$$

with $\|b_t\|_\infty \le k$, so

$$\text{s. c.} \Leftrightarrow \lim_{t\to 0} \|\varphi_t(z) - z\| = 0.$$

Hardy spaces H^p, $(1 \leq p < \infty)$, and Bergman spaces A^p_α, $(1 \leq p < \infty, -1 < \alpha < \infty)$, satisfy the above conditions, and $\|\varphi_t(z) - z\| \to 0$ is a consequence of Lebesgue dominated convergence.

Every $\{C_t\}$ is s. c. on H^p (Berkon - Porta), and on A^p_α.

2. Dirichlet spaces.

Every $\{C_t\}$ is s. c. on $\mathcal{D}$.

$\mathcal{D}$ is a member of $\mathcal{D}_\alpha$, $-1 < \alpha < \infty$,

$$\|f\|^2_{\mathcal{D}_\alpha} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha \, dxdy < \infty,$$

$\mathcal{D}_0 = \mathcal{D}$
$\mathcal{D}_1 = H^2$
$\mathcal{D}_\alpha =$ a Bergman space for $\alpha > 1$

Every $\{C_t\}$ is s. c. on $\mathcal{D}_\alpha$ for $0 < \alpha < 1$ (P. Galanopoulos)

3. $X = H^\infty$ or $X = \mathcal{B}$, the Bloch space.

Both H^∞ and $\mathcal{B}$,

• are *Grothendieck spaces*: every weak* convergent $(x_n) \subset X^*$ is also weakly convergent in X^*.

• have the *Dunford-Pettis property*: given any sequences

$$(x_n) \subset X, \quad (x_n^*) \subset X^*,$$

weakly convergent to zero, then $\langle x_n, x_n^* \rangle \to 0$.

By a theorem of H. Lotz, if X is such a space and $\{T_t\}$ **any** semigroup on X,

$$\{T_t\} \text{ is s. c.} \Rightarrow \{T_t\} \text{ is uniformly continuous.}$$

In particular any s. c. $\{C_t\}$ has a bounded generator

$$\Gamma(f) = Gf'$$

and it follows that $\{C_t\}$ is trivial.

There are no nontrivial s. c. $\{C_t\}$ on H^∞ or on $\mathcal{B}$.

4. $X = VMOA$ **or** $X = \mathcal{B}_0$**, the little Bloch.**

- Polynomials are dense in X
- $\sup_{0<t<\delta} \|C_t\|_{X\to X} < \infty$,

thus for s. c. it suffices

$$\lim_{t\to 0} \|f\circ\varphi_t - f\|_X = 0, \quad f \text{ polynomial.}$$

But then

$$f\circ\varphi_t - f \in \mathcal{D} \subset VMOA \subset \mathcal{B}_0,$$

and

$$\|f\circ\varphi_t - f\|_{\mathcal{B}} \leq C_1\|f\circ\varphi_t - f\|_* \leq C_2\|f\circ\varphi_t - f\|_{\mathcal{D}},$$

Every $\{C_t\}$ is s. c. on VMOA and on $\mathcal{B}_0$.

5. Some other Mobius invariant spaces.

Consider the Q_p spaces. Let $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$.

$$f \in Q_p \Leftrightarrow \sup_{a\in\mathbb{D}} \|f \circ \phi_a - f(a)\|_{\mathcal{D}_p} < \infty,$$

$$f \in Q_{p,0} \Leftrightarrow \lim_{|a|\to 1} \|f \circ \phi_a - f(a)\|_{\mathcal{D}_p} \to 0.$$

We have

$$Q_1 = BMOA, \quad Q_{1,0} = VMOA$$

$$Q_p = \mathcal{B}, \quad \text{and} \quad Q_{p,0} = \mathcal{B}_0, \quad \text{for } p > 1,$$

and

$$\mathcal{D} \subset (Q_{r,0}, Q_r) \subset (Q_{s,0}, Q_s) \subset (\mathcal{B}_0, \mathcal{B}),$$

for $0 < r < s < \infty$, with a norm domination inequality.

The argument used for $VMOA, \mathcal{B}_0$, implies:

Every $\{C_t\}$ is s. c. on each $Q_{p,0}$, $p > 0$. (K. Wirths - J. Xiao)

6. The disc Algebra $A(\mathbb{D})$.

Theorem. If $\{C_t\}$ "has the right" to be s. c. then it is s. c.

"has the right" means: $\varphi_t \in A(\mathbb{D})$, for all $t \geq 0$,

The proof is nontrivial, involves the Caratheodory extension theorem for conformal maps - prime ends.

Further,

$\{C_t\}$ s. c. $\Leftrightarrow \partial_\infty \Omega$ is locally connected,

$\Omega = h(\mathbb{D})$, and h the Königs map of φ_t, i.e.

$$h(\varphi_t(z)) = e^{ct} h(z), \quad \text{if} \;\; b \in \mathbb{D}$$

$$h(\varphi_t(z)) = h(z) + t, \quad \text{if} \;\; b \in \partial\mathbb{D}$$

(M. Contreras - S. Diaz-Madrigal)

7. Hardy space on the upper half-plane.

$f : \mathbb{U} \to \mathbb{C}$ is in $H^p(\mathbb{U})$ if

$$\|f\|_p^p = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dx < \infty.$$

In analogy to $\mathbb{D}$ consider semigroups $\{\psi_t\}$ in $\mathbb{U}$ and the induced $\{C_t\}$.

Theorem Let $1 \leq p < \infty$. If $\{C_t\}$ consists of bounded operators on $H^p(\mathbb{U})$ then $\{C_t\}$ is s. c. on $H^p(\mathbb{U})$.

If C_t is bounded for some $t \neq 0$, then bounded for all t. (A. Arvanitidis)

8. X = BMOA. Sarason (1975) proved:

If $f \in BMOA$ then these are equivalent:

(a) $f \in VMOA$.
(b) $\lim_{t\to 0} \|f(e^{it}z) - f(z)\|_* = 0$.
(c) $\lim_{t\to 0} \|f(e^{-t}z) - f(z)\|_* = 0$.

As a consequence,

$\varphi_t(z) = e^{it}z$, $\varphi_t(z) = e^{-t}z$ induce a $\{C_t\}$ which is not s. c. on BMOA.

$\{C_t\}$ not s. c. on any X with

$$VMOA \subsetneq X \subset BMOA$$

For more general $\{\varphi_t\}$?

Ex. 1. Take

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t}$$

and

$$f(z) = \log(\frac{1}{1-z}) \in BMOA \setminus VMOA.$$

Then

$$\lim_{t\to 0} \|f \circ \varphi_t - f\|_* = ... = \lim_{t\to 0} t = 0,$$

thus $\{\varphi_t\}$ induces a s. c. $\{C_t\}$ on

$$V = \mathrm{span}\{VMOA,\ \log(1/(1-z))\},$$

Ex. 2. Take a starlike univalent

$$h : \mathbb{D} \to \mathbb{C}, \quad h(0) = 0,$$

such that $h \in BMOA \setminus VMOA$, and let

$$\varphi_t(z) = h^{-1}(e^{-t} h(z)).$$

If $k \geq 1$ and $h^k \in BMOA$, then

$$\lim_{t \to 0} \|(h \circ \varphi_t)^k - h^k\|_* = \lim_{t \to 0} |e^{-kt} - 1| \|h^k\|_* = 0,$$

so again $\{C_t\}$ is s. c. on the larger space

$$V = \text{span}\{VMOA,\ h, h^2, \cdots h^k\}.$$

Def. Given a $\{\varphi_t\}$ write

$$[\varphi_t, BMOA]$$

for the maximal closed subspace of $BMOA$ such that $\{\varphi_t\}$ induces a s. c. $\{C_t\}$ on it.

Thus

$$VMOA \subseteq [\varphi_t, BMOA] \subseteq BMOA$$

for every $\{\varphi_t\}$, and,

- for some $\{\varphi_t\}$, $\quad VMOA \subsetneq [\varphi_t, BMOA]$.
- for other $\{\varphi_t\}$, $\; VMOA = [\varphi_t, BMOA]$.

Q. For which $\{\varphi_t\}$ is $VMOA = [\varphi_t, BMOA]$?

Conditions under which $VMOA = [\varphi_t, BMOA]$.

Recall, the generator of $\{\varphi_t\}$ has the form

$$G(z) = (\overline{b}z - 1)(z - b)F(z),$$

where $b \in \overline{\mathbb{D}}$ and $\operatorname{Re} F(z) \geq 0$. Now

$$|F(z)| \geq C\frac{1-|z|}{1+|z|}, \quad |z| \to 1$$

so if the DW point $b \in \mathbb{D}$,

$$\frac{1-|z|}{G(z)} = O(1), \quad |z| \to 1.$$

Theorem. Let G be the generator of $\{\varphi_t\}$. If for some $0 < \alpha < 1$,

we have

$$\frac{(1-|z|)^{\alpha}}{G(z)} = O\left(1\right), \quad |z| \to 1,$$

then $VMOA = [\varphi_t, BMOA]$.

Thus in addition to

$$\varphi_t(z) = e^{-t} z, \qquad \varphi_t(z) = e^{it} z,$$

there are plenty of $\{\varphi_t\}$ for which $VMOA = [\varphi_t, BMOA]$.

For example if $\{\varphi_t\}$ has generator

$$G(z) = -z(1-z)^{\alpha}, \quad 0 < \alpha < 1,$$

then $VMOA = [\varphi_t, BMOA]$. Many other can be constructed.

Necessary condition:

<u>Theorem.</u> If $b \in \mathbb{D}$ and if $VMOA = [\varphi_t, BMOA]$, then

$$\lim_{|z|\to 1} \frac{1-|z|}{G(z)} = 0.$$

(Blasco, Contreras, Diaz-Madrigal, Martinez, S.)

A characterization of $VMOA = [\varphi_t, BMOA]$.

Consider a $\{C_t\}$ on BMOA.

Each φ_t is univalent so C_t maps VMOA into itself. Thus

$$S_t = C_t|_{VMOA}$$

is s. c. on VMOA. Because of the duality

$$VMOA^{**} = BMOA,$$

we find

$$S_t^{**} = C_t.$$

Let Γ be the infinitesimal generator of $\{S_t\}$,

$$\Gamma(f)(z) = G(z)f'(z),$$

unbounded on VMOA.

For $\lambda \in \rho(\Gamma)$ we consider the resolvent operator

$$R(\lambda, \Gamma) : VMOA \to VMOA$$

Some arguments from the general theory of semigroups and the VMOA-BMOA duality give:

<u>Theorem.</u> The following are equivalent

(i) $VMOA = [\varphi_t, BMOA]$.
(ii) $R(\lambda, \Gamma)^{**}(BMOA) \subset VMOA$, (i.e. $R(\lambda, \Gamma)$ is weakly compact on VMOA).

A more concrete characterization

One description of $[\varphi_t, BMOA]$ is,

$$[\varphi_t, BMOA] = \overline{\{f \in BMOA : Gf' \in BMOA\}}$$

Now, heuristically,

$$\{f \in BMOA : Gf' \in BMOA\} = \{f \in BMOA : Gf' = m \in BMOA\}$$

and

$$\begin{aligned} Gf' = m \Leftrightarrow f(z) &= \int^z m(\zeta)\frac{1}{G(\zeta)}\, d\zeta \\ \Leftrightarrow f(z) &= \int^z m(\zeta)g'(\zeta)\, d\zeta \end{aligned}$$

where $g(z) = \int^z \frac{1}{G(\zeta)}\, d\zeta$, This brings into the picture the operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta.$$

for a specific symbol g, and then

$$[\varphi_t, BMOA] = \overline{BMOA \cap T_g(BMOA)}$$

In particular if

$$T_g(BMOA) \subset VMOA,$$

then

$$[\varphi_t, BMOA] = VMOA.$$

For T_g we know:

it is bounded on BMOA $\Leftrightarrow$ it is bounded on VMOA $\Leftrightarrow$

$$\sup_{I\in\partial\mathbb{D}}\left\{\frac{\left(\log\frac{2}{|I|}\right)^2}{|I|}\int_{R(I)}|g'(z)|^2(1-|z|^2)dm(z)\right\}<\infty. \qquad \text{(C1)}$$

And
it is compact on BMOA $\Leftrightarrow$ it is compact on VMOA $\Leftrightarrow$

$$\lim_{|I|\to 0}\{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}=0. \qquad \text{(C2)}$$

Furthermore,

<u>Theorem.</u> Suppose T_g is bounded on BMOA, then,

$$T_g(BMOA)\subset VMOA \ \Leftrightarrow\ T_g \text{ is compact}$$

Correct symbol for T_g:

Def. Given a $\{\varphi_t\}$ with generator G and DW point b,

$$\text{If} \quad b \in \mathbb{D} \quad \text{let} \quad g(z) = g_\varphi(z) = \int_b^z \frac{\zeta - b}{G(\zeta)} d\zeta,$$

$$\text{If} \quad b \in \partial\mathbb{D} \quad \text{let} \quad g(z) = g_\varphi(z) = \int_0^z \frac{1}{G(\zeta)} d\zeta.$$

Theorem. Given a $\{\varphi_t\}$, let $g = g_\varphi$ as above, and assume g satisfies (C1). Then

$$\begin{aligned} [\varphi_t, BMOA] = VMOA &\Leftrightarrow T_g(BMOA) \subset VMOA \\ &\Leftrightarrow T_g \text{ is compact} \\ &\Leftrightarrow g \text{ satisfies (C2)} \end{aligned}$$

Remark: there are $\{\varphi_t\}$ for which g does not satisfy (C1), i.e. T_g is not bounded on BMOA.

Bloch space.

In a similar way we have

- $[\varphi_t, \mathcal{B}] = \mathcal{B}_0 \Leftrightarrow R(\lambda, \Gamma)^{**}(\mathcal{B}) \subset \mathcal{B}_0$

- Suppose $g = g_\varphi$ satisfies

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\log\left(\frac{1}{1-|z|^2}\right)|g'(z)| < \infty \tag{C1-$\mathcal{B}$}$$

Then $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1}(1-|z|^2)\log\left(\frac{1}{1-|z|^2}\right)|g'(z)| = 0. \tag{C2-$\mathcal{B}$}$$

(Blasco, Contreras, Diaz-Madrigal, Martinez, Papadimitrakis, S.)

some questions

1. Are there $\{\varphi_t\}$ for which $[\varphi_t, BMOA] = BMOA$?

2. Are there $\{\varphi_t\}$ with DW point $b \in \partial\mathbb{D}$ such that $[\varphi_t, BMOA] = VMOA$?

3. For which $\{\varphi_t\}$ is $[\varphi_t, Q_p] = Q_{p,0}$?

4. Are there $\{\varphi_t\}$ for which $[\varphi_t, Q_p] = Q_p$

5. T_g weakly compact on $Q_{p,0} \Leftrightarrow T_g$ compact?

Being the last speaker,

I would like to thank Daniel and all organizers

for an excellent meeting

and express the hope that the next meeting

will be as good as this one.