UNIVERSIDAD
DE MÁLAGA

Escuela Politécnica Superior.
Departamento de Matemática Aplicada.
Programa de Doctorado en Ingeniería Mecánica y Eficiencia Energética.

TESIS DOCTORAL

Conexiones de Galois y Técnicas de Tratamiento de la Información.

Dª. Francisca García Pardo.

Directora: Dra. Dª. Inmaculada de las Peñas Cabrera.

Málaga, 2016.

Summary

Galois connections are ubiquitous; together with adjunctions, their close relatives, occur in a number of research areas, ranging from the most theoretical to the most applied. In a rather poetic tone, the preface of [19] reads, *Galois connections provide the structure-preserving passage between two worlds of our imagination*; and we should add that these two worlds can be so different that the slightest relationship could be seldom ever imagined.

The term *Galois connection* was coined by Øystein Ore [46] (originally, spelled *connexion*) as a general type of correspondence between structures, obviously named after the Galois theory of equations which is an example linking subgroups of automorphisms and subfields. Ore generalized to complete lattices the notion of *polarity*, introduced by Birkhoff [10] several years before, as a fundamental construction which leads from any binary relation to inverse dual isomorphisms. Later, when Kan introduced the *adjoint functors* [36] in a categorical setting, his construction was noticed to greatly resemble that of the Galois connection; actually, in some sense, both notions are interdefinable: an adjunction between A and B is a Galois connection in which the order relation on B is reversed (this leads to the use of the term *isotone Galois connection* to refer to an adjunction between ordered structures).

The importance of Galois connections/adjunctions quickly increased to an extent that, for instance, the interest of category theorists moved from

universal mapping properties and natural transformations to adjointness.

In recent years there has been a notable increase in the number of publications concerning Galois connections, both isotone and antitone. On the one hand, one can find lots of papers on theoretical developments or theoretical applications [14, 19, 38]. See [43] for a first survey on applications, although more specific references on certain topics can be found, for instance, to programming [45] or logic [35]. Likewise, one can find published works concerning Galois connection from a categorical point of view as [15, 29].

Last but not least, it is worth noting that many of these works use Galois connections for dealing with Formal Concept Analysis (FCA), either theoretically or applicatively, since the derivation operators used to define the concepts form a (antitone) Galois connection. In [20], one can find a general view of this relation. Bělohlávek and Konečný [8] stress on the "duality" between isotone and antitone Galois connections in showing a case of reducibility of the concept lattices generated by using each type of connection, in such a way that the "duality" just works one way; Valverde and Peláez have studied the extension of conceptualization modes in [51], and provided a general approach to the discipline; Díaz and Medina [21] use Galois connections as building blocks for solving the multi-adjoint relation equations.

In the fuzzy case, several papers on fuzzy Galois connections or fuzzy adjunctions have been written since its introduction by Bělohlávek in [3]; consider for instance [9, 28, 39, 52] for some recent generalizations. Some authors have introduced alternative approaches guided by the intended applications: for instance, Shi et al [50] introduced a definition of fuzzy adjunction for its use in fuzzy mathematical morphology. Our approach in this thesis is more in consonance with Bělohlávek's logic approach, but in terms of the generalization provided by Yao and Li [52] within the framework of fuzzy posets and fuzzy closure operators.

The ability to build or define a Galois connection between two ordered structures is a matter of major importance, and not only for FCA. For instance, [16] establishes a Galois connection between valued constraint languages and sets of weighted polymorphisms in order to develop an algebraic theory of complexity for valued constraint languages.

A number of results can be found in the literature concerning sufficient or necessary conditions for a Galois connection between ordered structures to exist. The main results of this thesis are related to the existence and construction of the adjoint pair to a given mapping f, but *in a more general framework*.

Our initial setting is to consider a mapping $f\colon A \to B$ from a partially ordered (resp. preordered) set A into an unstructured set B, and then, characterizing those situations in which the set B can be partially ordered (resp. preordered) and an isotone mapping $g\colon B \to A$ can be built such that the pair (f, g) is an adjunction. On the other hand, there exists a tight relation between adjunctions and closure systems, in that every adjunction (f, g) leads to a closure operator $g \circ f$ and every closure operator leads to a closure system. Conversely, from any closure operator, an adjunction can be defined. Therefore, after obtaining the necessary and sufficient conditions to define a preorder on B, it makes sense to express those conditions in terms of the corresponding closure system in a (pre-)ordered setting.

We finish this thesis with the extension to a fuzzy framework the different characterizations and results obtained in the crisp case. Specifically, we work with fuzzy adjunctions on crisp sets with fuzzy ordered relations (fuzzy partial ordering) and with fuzzy preordered relations (fuzzy preordering).

When examining the literature, one can notice a lack of uniformity in the use of the term Galois connection, mainly due to its close relation to adjunctions and that, furthermore, there are two versions of each one. In the Chapter 2, after recalling the different interpretations usually assigned to the term Galois connection, we study the different characterizations and properties of the notion of adjunction between preordered sets and the relation among then. Moreover, we show that all four types essentially coincide. Hence all the results of this thesis are stated in terms of adjunction, though all of them can be straightforwardly used for any of the four notions.

Section 2.3.1 focuses on the case in which the domain A of a mapping $f\colon A \to B$ is a partially ordered set and in Section 2.3.2 we tackle the study done in the previous section but in the preordered case, and it is worth to be remarked that the absence of antisymmetry makes the proof of the results much more involved. We also introduce several considerations about the uniqueness of the right adjoint providing a number of toy examples.

Once we have addressed the problem of defining an adjoint pair, we observe that the composition of the two components of an adjunction leads to a $\approx$-closure operator which is compatible as well with the kernel relation associated to the left adjoint. Furthermore, the existence of a $\approx_A$-compatible closure system turns out to be a sufficient condition. This result shows the convenience of considering $\approx$-closure systems in the study of adjunctions in more general carriers.

In [3, 5], Bělohlávek generalized the notion of Galois connection to the framework of fuzzy logic. For a complete residuated lattice L and two universes U, V, instead of the traditional powersets 2^U and 2^V, Bělohlávek considered the L-powersets L^U and L^V and defined a *fuzzy Galois connection* (or an L- Galois connection) between U and V.

There are other recent extensions such as the alternative definition of fuzzy Galois connection given by Yao in [52]. This new vision of fuzzy adjunctions (Galois connections) generalizes Bělohlávek´s definition. We will adopt Yao´s approach to the notion of fuzzy adjunction.

In this way, Chapter 3 studies the different characterizations and properties of fuzzy adjunctions between sets with a fuzzy (pre)ordering relation.

Moreover, we also analyze, given $f : \langle A, \rho_A\rangle \to B$ where $\langle A, \rho_A\rangle$ is a set with a fuzzy (pre)order, the necessary and sufficient conditions to define ρ_B, a fuzzy preorder in B, and a right adjoint $g\colon \langle B, \rho_B\rangle \to \langle A, \rho_A\rangle$ for the mapping f.

The results on sets with fuzzy preordering relation have more applicability since antisymmetry, in practice, is sometimes a too strong requirement; the study of this problem is particularly challenging since other previous results are stated in terms of the existence of maximum elements which are unique precisely because of antisymmetry, which is no longer available in a preordered setting

Finally, we introduce the notion of closure system and closure operator on crisp sets with fuzzy ordering relations (resp. fuzzy preordering relations), together with a number of results which allow to simplify the presentation of the construction of the right adjoint.

Detailed description of the content of the thesis

Now, we will show the main definitions and results of this work. We will preserve the organization of the full manuscript. In this summary, we do not include all the preliminaries that can be found in detail in the thesis.

Galois connections between preordered sets

We formulate the results in the most general framework of preordered sets, which are sets endowed with a reflexive and transitive binary relation. We study the different definitions of Galois connection between preordered set, their characterization and the relation among them.

Definition 2.1: Let $\mathbb{A} = \langle A, \lesssim_A\rangle$ and $\mathbb{B} = \langle B, \lesssim_B\rangle$ be preordered sets and consider two mappings $f : A \to B$ and $g : B \to A$. The pair (f, g) is called a[1]

- *Right Galois connection between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g) : \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$, if the following condition holds

  $$a \lesssim_A g(b) \text{ if and only if } b \lesssim_B f(a) \quad \text{for all } a \in A \text{ and } b \in B.$$

- *Left Galois connection between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g) : \mathbb{A} \leftharpoondown\!\!\rightharpoondown \mathbb{B}$, if the following condition holds

  $$g(b) \lesssim_A a \text{ if and only if } f(a) \lesssim_B b \quad \text{for all } a \in A \text{ and } b \in B.$$

- *Adjunction between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, if the following condition holds

  $$a \lesssim_A g(b) \text{ if and only if } f(a) \lesssim_B b \quad \text{for all } a \in A \text{ and } b \in B.$$

- *Co-adjunction between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g) : \mathbb{A} \rightleftharpoons \mathbb{B}$, if the following condition holds

  $$g(b) \lesssim_A a \text{ if and only if } b \lesssim_B f(a) \quad \text{for all } a \in A \text{ and } b \in B.$$

All of the previous notions can be seen in the literature, in fact, one can even find the same term applied to different notions of connection/adjunction. Although it is true that the four definitions are strongly related, they do not have exactly the same properties; hence, it makes sense to specifically describe what is the relation between the four notions stated above, together with their corresponding characterizations.

The following theorem states the existence of pairwise biunivocal correspondences between all the notions above. The transition between the two

[1]The arrow notation for the different versions is taken from [51].

types of adjunctions (connections) relies on using the opposite ordering in *both* preordered sets, whereas the transition between adjunctions to connections and vice versa relies on using the opposite ordering in *just one* of the preordered sets.

Theorem 2.1: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ and $\mathbb{B} = \langle B, \lesssim_B \rangle$ be preordered sets and consider two mappings $f : A \to B$ and $g : B \to A$. Then, the following conditions are equivalent

1. $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$
2. $(f, g) : \mathbb{A}^{op} \rightleftharpoons \mathbb{B}^{op}$
3. $(f, g) : \mathbb{A} \leftharpoonup\!\!\!\rightharpoonup \mathbb{B}^{op}$
4. $(f, g) : \mathbb{A}^{op} \leftharpoondown\!\!\!\rightharpoondown \mathbb{B}$

Observe that, as a direct consequence of this theorem, any property about adjunctions can be extended by duality to the other kind of connections.

Any preordered set $\langle A, \lesssim_A \rangle$ induces the *symmetric kernel* relation in A defined as $a_1 \approx_A a_2$ if and only if $a_1 \lesssim_A a_2$ and $a_2 \lesssim_A a_1$ for $a_1, a_2 \in A$.

The notions of maximum and minimum in a poset can be extended to preordered sets as follows: an element $a \in A$ is a *p-maximum* (*p-minimum* resp.) for a set $X \subseteq A$ if $a \in X$ and $x \lesssim_A a$ ($a \lesssim_A x$, resp.) for all $x \in X$. The set of p-maximum (p-minimum) elements of X will be denoted as p-$\max X$ (p-$\min X$, resp.). Observe that, in a preordered set, different elements can be p-maximum for a set X, but, in this case, $a_1, a_2 \in$ p-$\max X$ implies $a_1 \approx a_2$.

Given a preordered set $\langle A, \lesssim_A \rangle$ and $a \in A$, the *downward closure* $a^{\downarrow}$ of a is defined as $a^{\downarrow} = \{x \in A \mid x \lesssim_A a\}$ and the *upward closure* $a^{\uparrow}$ of a is defined as $a^{\uparrow} = \{x \in A \mid a \lesssim_A x\}$.

Taking into account the definitions, we introduce the characterizations of the notion of adjunction between preordered sets.

Theorem 2.2: Let $\mathbb{A} = \langle A, \lesssim_A \rangle, \mathbb{B} = \langle B, \lesssim_B \rangle$ be two preordered sets and consider two mappings $f : A \to B$ and $g : B \to A$. The following conditions are equivalent:

i) $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.

ii) f and g are isotone maps, $g \circ f$ is inflationary and $f \circ g$ is deflationary.

iii) $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.

iv) $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$.

v) f is isotone and $g(b) \in \text{p-}\max f^{-1}(b^{\downarrow})$ for all $b \in B$.

vi) g is isotone and $f(a) \in \text{p-}\min g^{-1}(a^{\uparrow})$ for all $a \in A$.

A number of characterizations for the different Galois connections and adjunctions are summarized in Table 1.

Section 2.1 ends with several theorems which provide properties about Galois connections, adjunction and co-adjunction between preordered sets.

Theorem 2.3: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ and $\mathbb{B} = \langle B, \lesssim_B \rangle$ be preordered sets and consider two mappings $f : A \to B$ and $g : B \to A$. If $(f, g) \colon \mathbb{A} \Leftrightarrow \mathbb{B}$, where $\Leftrightarrow \in \{\leftharpoonup, \rightharpoondown, \leftrightharpoons, \rightleftharpoons\}$, then, $(f \circ g \circ f)(a) \approx_B f(a)$, for all $a \in A$, and $(g \circ f \circ g)(b) \approx_A g(b)$ for all $b \in B$. Moreover,

1. If (f, g) is both an adjunction and a co-adjunction (left and right Galois connection resp.) then $(g \circ f)(a) \approx_A a$ for all $a \in A$ and $(f \circ g)(b) \approx_B b$ for all $b \in B$.

2. If (f, g) is both a (left or right) Galois connection and a (co-) adjunction then $f(a_1) \approx_B f(a_2)$ for all $a_1, a_2 \in A$ with $a_1 \lesssim_A a_2$, and $g(b_1) \approx_B g(b_2)$ for all $b_1, b_2 \in B$ with $b_1 \lesssim_B b_2$.

Table 1: Galois connections and adjunctions: equivalent characterizations

Galois Connections	
Right Galois Connections between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \rightharpoonup \mathbb{B}$	Left Galois Connections between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftharpoondown \mathbb{B}$
$b \leq f(a) \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$f(a) \leq b \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are antitone and $g \circ f$ and $f \circ g$ are inflationary	f and g are antitone and $g \circ f$ and $f \circ g$ are deflationary
$f(a)^{\downarrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$	$f(a)^{\uparrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$
$g(b)^{\downarrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$	$g(b)^{\uparrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$
f is antitone and $g(b) \in \text{p-max}\, f^{-1}(b^{\uparrow})$ for all $b \in B$	f is antitone and $g(b) \in \text{p-min}\, f^{-1}(b^{\downarrow})$ for all $b \in B$
g is antitone and $f(a) \in \text{p-max}\, g^{-1}(a^{\uparrow})$ for all $a \in A$	g is antitone and $f(a) \in \text{p-min}\, g^{-1}(a^{\downarrow})$ for all $a \in A$
Adjunctions	
Adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$	Co-adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \rightleftharpoons \mathbb{B}$
$f(a) \leq b \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$b \leq f(a) \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are isotone, $g \circ f$ is inflationary and $f \circ g$ is deflationary	f and g are isotone, $g \circ f$ is deflationary and $f \circ g$ is inflationary
$f(a) \uparrow= g^{-1}(a^{\uparrow})$ for all $a \in A$	$f(a)^{\downarrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$
$g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$	$g(b)^{\uparrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$
f is isotone and $g(b) \in \mathrm{maxf}^{-1}(\mathrm{b}^{\downarrow})$ for all $b \in B$	f is isotone and $g(b) \in \min f^{-1}(b^{\uparrow})$ for all $b \in B$
g isotone and $f(a) \in \min g^{-1}(a^{\uparrow})$ for all $a \in A$	g is isotone and $f(a) \in \mathrm{maxg}^{-1}(\mathrm{a}^{\downarrow})$ for all $a \in A$

For any preordered set $\mathbb{A} = \langle A, \lesssim_A \rangle$, the quotient set over the symmetric kernel relation $\approx_A$ is denoted as $\overline{A}$. The relation defined as "$[a_1]_\approx \lesssim_{\overline{A}} [a_2]_\approx$ if and only if $a_1 \lesssim_A a_2$" is a partial order. The quotient posets $\langle \overline{A}, \lesssim_{\overline{A}} \rangle$ is denoted as $\overline{\mathbb{A}}$. Theorem 2.2 allows to translate adjunctions to the quotient posets as follows.

Given $\mathbb{A}$ and $\mathbb{B}$ two preordered sets and $f\colon \mathbb{A} \to \mathbb{B}$ an isotone (resp. antitone) mapping, we define a mapping $\overline{f}\colon \overline{\mathbb{A}} \to \overline{\mathbb{B}}$ where $\overline{f}([a]_\approx) = [f(a)]_\approx$.

Theorem 2.4: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ and $\mathbb{B} = \langle B, \lesssim_B \rangle$ be two preordered sets and consider $\overset{\leftharpoondown}{\rightharpoondown} \in \{\leftharpoonup\!\!\rightharpoonup, \leftharpoondown\!\!\rightharpoondown, \leftrightharpoons, \rightleftharpoons\}$. If $(f, g) : \mathbb{A} \overset{\leftharpoondown}{\rightharpoondown} \mathbb{B}$ then $(\overline{f}, \overline{g})\colon \overline{\mathbb{A}} \overset{\leftharpoondown}{\rightharpoondown} \overline{\mathbb{B}}$.

Corollary 2.1: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ and $\mathbb{B} = \langle B, \lesssim_B \rangle$ be two preordered sets and consider two mappings $f : A \to B$ and $g : B \to A$.

1. (f, g) is both an adjunction and a co-adjunction (a left Galois connection and a right Galois connection, resp.) if and only if f and g are isotone (resp. antitone) mappings and $\overline{f}$ and $\overline{g}$ are inverse mappings (i.e. $(\overline{f})^{-1} = \overline{g}$).

2. Both relations $\lesssim_A$ and $\lesssim_B$ are equivalence relations and (f, g) is an adjunction (resp. co-adjunction, right Galois connection, left Galois connection) if and only if (f, g) is adjunction, co-adjunction, right Galois connection and left Galois connection.

Construction of adjunctions between posets

Given $f\colon A \to B$ we first focus on the case in which the domain A is a partially ordered set and, once introduced the preliminary technical results, we provide the necessary and sufficient conditions for the existence of an ordering relation on B and a mapping $g\colon B \to A$ such that (f, g) constitutes an adjunction.

In general, given a poset $\langle A, \leq_A \rangle$ together with an equivalence relation $\sim$ on A, it is common to denote the quotient set of A wrt $\sim$ as $A_\sim = A/{\sim}$ and the natural projection $\pi\colon A \to A_\sim$. The equivalence class of an element $a \in A$ is denoted by $[a]_\sim$ and, then, $\pi(a) = [a]_\sim$.

With the aim of finding conditions for building a right adjoint to a mapping f from a poset $\langle A, \leq_A \rangle$ to an unstructured set B, we naturally consider the canonical decomposition of $f\colon A \to B$ through $A_{\equiv_f}$, the quotient set of A wrt the *kernel relation* $\equiv_f$ defined as $a \equiv_f b$ if and only if $f(a) = f(b)$ (see Figure 1). We denote the inclusion mapping by $i : f(A) \to B$ where $i(b) = b$ and $\varphi\colon A_{\equiv_f} \to f(A)$ is the unique bijective mapping which makes the following diagram commutative, i.e., $\varphi([a]_{\equiv_f}) = f(a)$

$$\begin{tikzcd}
\langle A, \leq_A \rangle \arrow[r, "f"] \arrow[d, dashed, two heads, "\pi"'] & B \\
A_{\equiv_f} \arrow[r, dashed, "\varphi"'] & f(A) \arrow[u, dashed, hook, "i"']
\end{tikzcd}$$

Figure 1: Canonical decomposition of $f\colon \langle A, \leq_A \rangle \to B$ through $A_{\equiv_f}$.

The following lemma provides sufficient conditions for the natural projection being the left component of an adjunction.

Lemma 2.2: Let $\langle A, \leq_A \rangle$ be a poset and $\sim$ an equivalence relation on A. Assume that the following conditions hold

1. there exists $\max([\mathrm{a}]_\sim)$, for all $a \in A$.
2. if $a_1 \leq_A a_2$ then $\max([\mathrm{a}_1]_\sim) \leq_\mathrm{A} \max([\mathrm{a}_2]_\sim)$, for all $a_1, a_2 \in A$.

Then, the relation $\leq_{A_\sim}$ defined by $[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$ if and only if $a_1 \leq_A \max([\mathrm{a}_2]_\sim)$ is an ordering in $A_\sim$ and, moreover, the pair $(\pi, \max)$ is an adjunction between $\langle A, \leq_A \rangle$ and $\langle A_\sim, \leq_{A_\sim} \rangle$.

The following result states that the conditions given in the previous Lemma are also necessary and that the ordering relation and the right adjoint are uniquely defined.

Lemma 2.3: Let $\langle A, \leq_A \rangle$ be a poset and $\sim$ an equivalence relation on A. Let $A_\sim = A/{\sim}$ be the quotient set of A wrt $\sim$ and $\pi \colon A \to A_\sim$ the natural projection. If there exists an ordering relation $\leq_{A_\sim}$ in $A_\sim$ and a mapping $g \colon A_\sim \to A$ such that $(\pi, g) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle A_\sim, \leq_{A_\sim} \rangle$ then,

1. $g([a]_\sim) = \max([\mathrm{a}]_\sim)$ for all $a \in A$.
2. $[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$ if and only if $a_1 \leq_A \max([\mathrm{a}_2]_\sim)$ for all $a_1, a_2 \in A$.
3. if $a_1 \leq_A a_2$ then $\max([\mathrm{a}_1]_\sim) \leq_\mathrm{A} \max([\mathrm{a}_2]_\sim)$ for all $a_1, a_2 \in A$.

We continue with the analysis of the canonical decomposition which, naturally, leads to the following result.

Lemma 2.4: Consider a poset $\langle A, \leq_A \rangle$ and a bijective mapping $\varphi \colon A \to B$, then there exists a unique ordering relation in B, which is defined as $b_1 \leq_B b_2$ if and only if $\varphi^{-1}(b_1) \leq_A \varphi^{-1}(b_2)$, such that $(\varphi, \varphi^{-1}) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle B, \leq_B \rangle$.

As a consequence of the previous results, we have established the necessary and sufficient conditions to ensure the existence and uniqueness of a right adjoint for any *surjective* mapping f from a poset A to an unstructured set B.

Theorem 2.5: Given a poset $\langle A, \leq_A \rangle$ and a surjective mapping $f \colon A \to B$, let $\equiv_f$ be the kernel relation. Then, there exists an ordering $\leq_B$ in B and a mapping $g \colon B \to A$ such that $(f, g) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle B, \leq_B \rangle$ if and only if

1. there exists $\max([a]_{\equiv_f})$ for all $a \in A$.
2. $a_1 \leq_A a_2$ implies $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$, for all $a_1, a_2 \in A$.

A summary of the construction of an adjunction, with f a surjective mapping, is represented in the Figure 2.

$$
\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, two heads, shift right, "\pi"'] & B \arrow[l, dashed, bend right, "g=\max\circ\varphi^{-1}"'] \arrow[dl, dashed, shift left, "\varphi^{-1}"] \\
A_{\equiv_f} \arrow[u, dashed, shift right, "\max"'] \arrow[ur, shift left, "\varphi"]
\end{tikzcd}
$$

Figure 2: (f, g) is an adjunction where f is surjective and $g = \max \circ\ \varphi^{-1}$.

Now, we tackle the same problem in the case of f being not necessarily surjective. Now, there are several possible orderings on B which allows us to define the right adjoint. The crux of the construction is related to the definition of an order-embedding of the image into the codomain set.

More generally, the idea is to extend an ordering defined just on a subset of a set to the whole set.

Given a subset $X \subseteq B$, and a fixed element $m \in X$, any preordering $\leq_X$ in X can be extended to a preordering $\leq_m$ on B, defined as the reflexive and transitive closure of the relation $\leq_X \cup \{(m, y) \mid y \notin X\}$. Note that the relation above can be described, for all $x, y \in B$, as $x \leq_m y$ if and only if some of the following conditions holds:

(a) $x, y \in X$ and $x \leq_X y$

(b) $x \in X, y \notin X$ and $x \leq_X m$

(c) $x, y \notin X$ and $x = y$

If the relation $\leq_X$ in X is an ordering then any extension $\leq_m$ on B is antisymmetric as well.

Lemma 2.5: Given a subset $X \subseteq B$, and a fixed element $m \in X$, then $\leq_X$ is an ordering on X if and only if $\leq_m$ is an ordering on B.

Lemma 2.6: Let X be a subset of B, consider a fixed element $m \in X$, and an ordering relation $\leq_X$ in X. Define the mapping $j_m\colon \langle B, \leq_m \rangle \to \langle X, \leq_X \rangle$ as

$$j_m(x) = \begin{cases} x & \text{if } x \in X \\ m & \text{if } x \notin X \end{cases}$$

Then, (i, j_m) constitutes an adjunction between $\langle X, \leq_X \rangle$ and $\langle B, \leq_m \rangle$, where i denotes the inclusion $X \hookrightarrow B$.

From the last results, we obtain one of the main theorem of Section 2.3.1 which shows the necessary and sufficient conditions to define a suitable ordering relation on B and a mapping $g\colon B \to A$ such that (f, g) forms an adjunction between ordered sets.

Theorem 2.6: Given a poset $\langle A, \leq_A \rangle$ and a mapping $f\colon A \to B$, let $\equiv_f$ be the kernel relation. Then, there exists an ordering $\leq_B$ in B and a mapping $g\colon B \to A$ such that $(f, g)\colon \langle A, \leq_A \rangle \leftrightharpoons \langle B, \leq_B \rangle$ if and only if

1. there exists $\max([a]_{\equiv_f})$ for all $a \in A$.
2. $a_1 \leq_A a_2$ implies $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$, for all $a_1, a_2 \in A$.

Pictorially, the mapping g above is the composition $\max \circ \varphi^{-1} \circ j_m$ (see Figure 3). By Theorem 2.5, there exists an ordering $\leq_{f(A)}$ on $f(A)$ and, considering an arbitrary element $m \in f(A)$, the ordering $\leq_{f(A)}$ induces an ordering $\leq_m$ on B, as stated in Lemma 2.5.

Construction of adjunctions between preordered sets

We also extend the analogous construction to the framework of preordered sets. The idea underlying the construction is similar to that above, but the absence of antisymmetry makes the low level computations much more involved than in the partially ordered case.

$$
\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, shift left, two heads, "\pi"] & B \arrow[l, bend right, dashed, "g=\max\circ\varphi^{-1}\circ j_m"'] \arrow[d, shift left, dashed, "j_m"] \\
A_{\equiv_f} \arrow[u, shift left, dashed, "\max"] \arrow[r, shift left, "\varphi"] & f(A) \arrow[u, shift left, hook, "i"] \arrow[l, shift left, dashed, "\varphi^{-1}"]
\end{tikzcd}
$$

Figure 3: (f, g) is an adjunction where $g = \max \circ\ \varphi^{-1} \circ j_m$.

In order to study the existence of adjoints in this framework, we need to use the previously defined relation $\approx_A$ and we will keep using the kernel relation $\equiv_f$. The two relations above are used together in the definition of the *p-kernel* relation defined below:

Definition 2.2: Let $\langle A, \lesssim_A \rangle$ be a preordered set and consider a mapping $f\colon A \to B$. The *p-kernel* relation $\cong_A$ on A is the equivalence relation obtained as the transitive closure of the union of the symmetric kernel relation $\approx_A$ and kernel relation $\equiv_f$.

The following definition recalls the Hoare preordering between subsets of a preordered set and introduces the notion of cyclic subset.

Definition 2.3: Let $\langle A, \lesssim_A \rangle$ be a preordered set, and consider $X, Y \subseteq A$.

- $X \sqsubseteq_H Y$ if and only if, for all $x \in X$, there exists $y \in Y$ such that $x \lesssim_A y$. This is called *Hoare relation*.
- X is said to be *cyclic* if $x \approx_A y$ for all $x, y \in X$.

An alternative characterization for the Hoare preorder is provided for the case of cyclic subsets.

Lemma 2.7: Let $\langle A, \lesssim_A \rangle$ be a preordered set and $X, Y \subseteq A$ non-empty subsets, where Y is cyclic. Then, the following statements are equivalent:

1. $X \sqsubseteq_H Y$.

2. There exist $x \in X$ and $y \in Y$ such that $x \lesssim_A y$.

3. $x \lesssim_A y$, for all $x \in X$ and $y \in Y$.

Let $\langle A, \lesssim_A \rangle$ be a preordered set and let X be a subset of A. The set of upper bounds of X is defined as follows

$$\mathrm{UB}(X) = \{b \in A \mid x \lesssim_A b \text{ for all } x \in X\}$$

For the construction, given a mapping $f \colon \langle A, \lesssim_A \rangle \to B$ from a preordered set $\langle A, \lesssim_A \rangle$ to an unstructured set B, our first goal is to find sufficient conditions to define a suitable preordering on B such that a right adjoint exists, in the style of Lemma 2.2. Notice that there is much more than a mere adaptation of the result for posets.

Lemma 2.8: Let $\langle A, \lesssim_A \rangle$ be a preordered set and consider a surjective mapping $f \colon \langle A, \lesssim_A \rangle \to B$. Consider $S \subseteq A$ such that the following conditions hold:

- $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
- $\text{p-min}(\mathrm{UB}([a]_{\cong_A}) \cap S) \neq \varnothing$, for all $a \in A$.
- If $a_1 \lesssim_A a_2$, then $\text{p-min}(\mathrm{UB}([a_1]_{\cong_A}) \cap S) \sqsubseteq_H \text{p-min}(\mathrm{UB}([a_2]_{\cong_A}) \cap S)$.

Then, there exists a preorder $\lesssim_B$ in B and a map g such that $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$.

If the initial mapping f is not surjective, we have established the conditions to construct a right adjoint g' for the restriction of f to its image set and then, as the following Lemma shows, an adequate extension of g' to B provides a right adjoint for the initial f.

Lemma 2.9: Consider a preordered set $\langle A, \lesssim_A \rangle$, a set B and a mapping $f\colon A \to B$. Then, there exist a preorder $\lesssim_B$ on B and an adjunction $(f, g)\colon \langle A, \lesssim_A \rangle \leftrightharpoons \langle B, \lesssim_B \rangle$ if and only if there exist a preorder $\lesssim_{f(A)}$ on $f(A)$ and an adjunction $(f, g')\colon \langle A, \lesssim_A \rangle \leftrightharpoons \langle f(A), \lesssim_{f(A)} \rangle$.

It can be shown that the pair $(f, g)\colon \langle A, \lesssim_A \rangle \leftrightharpoons \langle B, \lesssim_B \rangle$ is an adjunction, where g is the extension of g' defined by

$$g(x) = \begin{cases} g'(x) & \text{if } x \in f(A) \\ g'(m) & \text{if } x \notin f(A) \end{cases}$$

where $m \in f(A)$.

The main result in this section is the corresponding version of Theorem 2.6, which is a twofold extension of the statement of Lemma 2.8 in that, firstly, the mapping f need not be surjective and, secondly, it gives the necessary and sufficient conditions for the existence of an adjunction between preordered sets.

Theorem 2.7: Given any preordered set $\mathbb{A} = \langle A, \lesssim_A \rangle$ and a mapping $f\colon \mathbb{A} \to B$, there exists a preorder $\mathbb{B} = \langle B, \lesssim_B \rangle$ and $g\colon B \to A$ such that $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ if and only if there exists a subset S of A such that the following conditions hold:

1. $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(\text{UB}([a]_{\cong_A}) \cap S) \neq \varnothing$, for all $a \in A$.
3. If $a_1 \lesssim_A a_2$, then $\text{p-min}(\text{UB}([a_1]_{\cong_A}) \cap S) \sqsubseteq_H \text{p-min}(\text{UB}([a_2]_{\cong_A}) \cap S)$.

We finish with several considerations on the uniqueness of right adjoints and the induced ordered structure in the codomains. It is well-known that given two posets $\mathbb{A} = \langle A, \leq_A \rangle$ and $\mathbb{B} = \langle B, \leq_B \rangle$ and a mapping $f\colon A \to B$, if there exists $g: B \to A$ such that the pair (f, g) is an adjunction, then it is unique.

This uniqueness property has been extended, in the case of surjective mappings, not only to the right adjoint, but also to the ordering relation in the codomain: namely, there exists just one partial ordering on the codomain B such that a right adjoint exists.

Contrariwise to the partially ordered case, given two preordered sets $\mathbb{A} = \langle A, \lesssim_A \rangle$ and $\mathbb{B} = \langle B, \lesssim_B \rangle$ and a mapping $f: A \to B$, the unicity of the mapping $g: B \to A$ satisfying $(f, g): \mathbb{A} \leftrightharpoons \mathbb{B}$, when it exists, cannot be guaranteed. But if g_1 and g_2 are right adjoints, then $g_1(b) \approx_A g_2(b)$ for all $b \in B$, and one usually says that the right adjoint is *essentially unique*. However, and this is the interesting part, the unicity of the ordering cannot be extended in general in the preordered case when the codomain is unstructured.

Adjunctions and closure systems on preordered sets

In this section, we state the necessary and sufficient conditions obtained in the previous section in terms of closure operators and closure systems. Closure operators and closure systems are different approaches to the same phenomenon. We focus now on the development of the well-known link between these two notions on a partially ordered set, but in the more general framework of preordered sets.

To begin with, both notions have to be adapted to the lack of antisymmetry. This involves the use of the symmetric kernel relation $\approx$ introduced in the previous sections.

Definition 2.6: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ be a preordered set.

1. A mapping $c: A \to A$ is said to be a $\approx_A$-closure operator if c is inflationary, isotone and $\approx_A$-idempotent, i.e. $(c \circ c)(a) \approx_A c(a)$, for all $a \in A$.

2. A subset $S \subseteq A$ is a $\approx_A$-closure system if the set p-$\min(a^\uparrow \cap S)$ is non-empty for all $a \in A$.

The notion of $\approx_A$-closed set can be found in [24], whereas the previous version of $\approx_A$-closure system is, to the best of our knowledge, a novel notion.

It is convenient to introduce the notion of compatibility with an equivalence relation.

Definition 2.8: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ be a preordered set and consider an equivalence relation $\sim$ on A.

1. A $\approx_A$-closure operator $c\colon A \to A$ is said to be *compatible with respect to* $\sim$ if $a \sim b$ implies $c(a) \approx_A c(b)$ for all $a, b \in A$.
2. Similarly, a $\approx_A$-closure system S is said to be *compatible with respect to* $\sim$ if $a \lesssim_A s$ implies $[a]_\sim \subseteq s^\downarrow$, for all $a \in A, s \in S$.

The notion of compatibility in the previous definition is preserved when moving between operators and systems. This is formally stated in the following result:

Lemma 2.10: Let $c\colon A \to A$ be a $\approx_A$-closure operator compatible wrt an equivalence relation $\sim$ on A, then the $\approx_A$-closure system $S_c = \{x \in A \mid c(a) = a\}$ is compatible wrt $\sim$.

Conversely, let S be a $\approx_A$-closure system compatible wrt $\sim$, then any $\approx_A$-closure operator c associated to S is compatible wrt $\sim$ as well.

Lemma 2.11: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ be a preordered set and consider a mapping $f\colon \mathbb{A} \to B$. A $\approx_A$-closure system is compatible wrt $\equiv_f$ if and only if it is compatible wrt $\cong_A$.

We state that the composition of the two components of the adjunction leads to a $\approx_A$-closure operator which, moreover, is compatible wrt the kernel relation associated to f. As a result, the existence of a $\approx_A$-compatible system turns out to be a necessary condition. The following main result states that this condition is also sufficient.

Theorem 2.9: Let $\mathbb{A} = \langle A, \lesssim_A \rangle$ be a preordered set and consider a mapping $f\colon \mathbb{A} \to B$. Then, there exists a preorder in B and a mapping $g\colon B \to A$ such that (f, g) forms an adjunction if and only if there exists a $\approx_A$-closure system S compatible wrt $\equiv_f$.

Adjunctions between fuzzy preordered sets

We devote this Section to establish the definitions and characterizations of fuzzy Galois connections and fuzzy adjunctions between sets with a fuzzy preorder. Moreover, we study the relations between them, their characterizations and properties. We will work with complete residuated lattices, $\mathbb{L} = (L, \leq, \top, \bot, \otimes, \to)$, as underlying structure for considering fuzziness.

Definition 3.1:

- An $\mathbb{L}$-*fuzzy preordered set* is a pair $\langle A, \rho_A \rangle$ in which ρ_A is a reflexive and transitive $\mathbb{L}$-fuzzy relation, i.e. $\rho_A(a, a) = \top$ and $\rho_A(a, b) \otimes \rho_A(b, c) \leq \rho_A(a, c)$ for all $a, b, c \in A$.
- An $\mathbb{L}$-*fuzzy ordered set* is a pair $\langle A, \rho_A \rangle$ in which ρ_A is a reflexive, transitive and antisymmetric $\mathbb{L}$-fuzzy relation, i.e. $\rho_A(a, b) = \rho_A(b, a) = \top$ implies $a = b$ for all $a, b \in A$.

From now on, we will omit the prefix $\mathbb{L}$.

Definition 3.3: Let $\mathbb{A} = \langle A, \rho_A \rangle$, $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preordered sets and consider two mappings $f : A \to B$ and $g : B \to A$. The pair (f, g) is said to be a

- *Right fuzzy Galois connection between* $\mathbb{A}$ *and* $\mathbb{B}$ and denoted by $(f, g) : \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$, if

$$\rho_A(a, g(b)) = \rho_B(b, f(a)) \quad \text{for all } a \in A \text{ and } b \in B.$$

- *Left fuzzy Galois connection between* $\mathbb{A}$ *and* $\mathbb{B}$ and denoted by $(f, g) : \mathbb{A} \leftharpoondown\!\!\rightharpoondown \mathbb{B}$, if

$$\rho_A(g(b), a) = \rho_B(f(a), b) \quad \text{for all } a \in A \text{ and } b \in B.$$

- *Fuzzy adjunction between* $\mathbb{A}$ *and* $\mathbb{B}$ and denoted by $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, if

$$\rho_A(a, g(b)) = \rho_B(f(a), b) \quad \text{for all } a \in A \text{ and } b \in B.$$

- *Fuzzy co-adjunction between* $\mathbb{A}$ *and* $\mathbb{B}$ and denoted by $(f, g) : \mathbb{A} \rightleftharpoons \mathbb{B}$, if

$$\rho_A(g(b), a) = \rho_B(b, f(a)) \quad \text{for all } a \in A \text{ and } b \in B.$$

Given a fuzzy poset $\langle A, \rho_A \rangle$, for every element $a \in A$, the extension to the fuzzy setting of the notions of upward closure and downward closure of the element a are defined by $a^{\uparrow}, a^{\downarrow} : A \to L$ where $a^{\uparrow}(u) = \rho_A(a, u)$ and $a^{\downarrow}(u) = \rho_A(u, a)$ for all $u \in A$. An element $a \in A$ is a *maximum* for a fuzzy set X if $X(a) = \top$ and $X \subseteq a^{\downarrow}$. The definition of *minimum* is similar.

On fuzzy preordered sets, due to the absence of antisymmetry, there exists a crisp set of maxima (resp. minima) for X, not necessarily a singleton, which we will denote *p-max(X)* (resp., *p-min(X)*).

From now on, we will use the following notation: for a mapping $f : A \to B$ and a fuzzy subset Y of B, the fuzzy set $f^{-1}(Y)$ is defined as $f^{-1}(Y)(a) = Y(f(a))$, for all $a \in A$.

Theorem 3.1: Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preordered sets and consider two mappings $f : A \to B$ and $g : B \to A$. The following conditions are equivalent:

1. $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$.

2. f and g are isotone, $g \circ f$ is inflationary and $f \circ g$ is deflationary.

3. $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.

4. $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$.

5. f is isotone and $g(b) \in$ p-$\max f^{-1}(b^{\downarrow})$ for all $b \in B$.

6. g is isotone and $f(a) \in$ p-$\min g^{-1}(a^{\uparrow})$ for all $a \in A$.

From the last definitions and theorem, we obtain characterizations for the cases of fuzzy Galois connections, fuzzy adjunction and fuzzy co-adjunction as summarized in Table 2.

Any fuzzy preordered set $\mathbb{A} = \langle A, \rho_A \rangle$ defines a (crisp) preordered set $\mathbb{A}_c = \langle A, \lesssim_A \rangle$ where $a \lesssim_A b$ iff $\rho_A(a, b) = \top$.

Lemma 3.1: Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preordered sets and consider two mappings $f\colon A \to B$ and $g\colon B \to A$. For $\Leftrightarrow \in \{\leftharpoonup\!\!\rightharpoonup, \leftharpoondown\!\!\rightharpoondown, \leftrightharpoons, \rightleftharpoons\}$, if $(f, g) : \mathbb{A} \Leftrightarrow \mathbb{B}$ then $(f, g)\colon \mathbb{A}_c \Leftrightarrow \mathbb{B}_c$

From a fuzzy preordered set $\mathbb{A} = \langle A, \rho_A \rangle$, by defining $\approx$ as the crisp equivalence relation $a \approx b$ if and only if $\rho_A(a, b) = \rho_A(b, a) = \top$, the quotient set $A/_{\approx}$ is a fuzzy poset with respect to the fuzzy binary relation $\rho_{A_{\approx}}$ defined by $\rho_{A_{\approx}}([a], [b]) = \rho_A(a, b)$. Moreover, any mapping f between fuzzy preordered sets defines a mapping $f_{\approx}$ over those quotient posets in the same way as in Theorem 2.4.

Theorem 3.2: Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preordered sets and consider two mappings $f\colon A \to B$ and $g\colon B \to A$. Then, for $\Leftrightarrow \in \{\leftharpoonup\!\!\rightharpoonup, \leftharpoondown\!\!\rightharpoondown, \leftrightharpoons, \rightleftharpoons\}$, $(f, g) : \mathbb{A} \Leftrightarrow \mathbb{B}$ if and only if $(f_{\approx}, g_{\approx})\colon \mathbb{A}/_{\approx} \Leftrightarrow \mathbb{B}/_{\approx}$.

Theorem 3.3: Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preordered sets and $\Leftrightarrow \in \{\leftharpoonup\!\!\rightharpoonup, \leftharpoondown\!\!\rightharpoondown, \leftrightharpoons, \rightleftharpoons\}$. If $(f, g)\colon \mathbb{A} \Leftrightarrow \mathbb{B}$ then, for all $a \in A, b \in B$, the

Table 2: Summary of definitions and equivalent characterizations

Fuzzy Galois connections	
Right fuzzy Galois connection between $\mathbb{A}$ and $\mathbb{B}$ $(f, g) \colon \mathbb{A} = \langle A, \rho_A \rangle \leftharpoondown\!\!\rightharpoonup \mathbb{B} = \langle B, \rho_B \rangle$	Left fuzzy Galois connection between $\mathbb{A}$ and $\mathbb{B}$ $(f, g) \colon \mathbb{A} = \langle A, \rho_A \rangle \leftharpoonup\!\!\rightharpoondown \mathbb{B} = \langle B, \rho_B \rangle$
$\rho_B(b, f(a)) = \rho_A(a, g(b))$ for all $a \in A$ and $b \in B$	$\rho_B(f(a), b) = \rho_A(g(b), a)$ for all $a \in A$ and $b \in B$
f and g are antitone maps and $g \circ f$ and $f \circ g$ are inflationary maps	f and g are antitone maps and $g \circ f$ y $f \circ g$ are deflationary maps
$f(a)^{\downarrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$	$f(a)^{\uparrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$
$g(b)^{\downarrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$	$g(b)^{\uparrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$
f is an antitone map and $g(b) \in \text{p-max}\, f^{-1}(b^{\uparrow})$ for all $b \in B$	f is an antitone map and $g(b) \in \text{p-min}\, f^{-1}(b^{\downarrow})$ for all $b \in B$
g is an antitone map and $f(a) \in \text{p-max}\, g^{-1}(a^{\uparrow})$ for all $a \in A$	g is an antitone map and $f(a) \in \text{p-min}\, g^{-1}(a^{\downarrow})$ for all $a \in A$
Fuzzy adjunction and fuzzy co-adjunction	
Fuzzy adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f, g) \colon \mathbb{A} = \langle A, \rho_A \rangle \leftrightarrows \mathbb{B} = \langle B, \rho_B \rangle$	Fuzzy co-adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f, g) \colon \mathbb{A} = \langle A, \rho_A \rangle \rightleftarrows \mathbb{B} = \langle B, \rho_B \rangle$
$\rho_B(f(a), b) = \rho_A(a, g(b))$ for all $a \in A$ and $b \in B$	$\rho_B(b, f(a)) = \rho_A(g(b), a)$ for all $a \in A$ and $b \in B$
f and g are isotone maps, $g \circ f$ is inflationary and $f \circ g$ is deflationary	f and g are isotone maps, $g \circ f$ is deflationary and $f \circ g$ is inflationary
$f(a) \uparrow = g^{-1}(a^{\uparrow})$ for all $a \in A$	$f(a)^{\downarrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$
$g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$	$g(b)^{\uparrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$
f is an isotone map and $g(b) \in \text{p-max}\, f^{-1}(b^{\downarrow})$ for all $b \in B$	f is an isotone map and $g(b) \in \text{p-min}\, f^{-1}(b^{\uparrow})$ for all $b \in B$
g is an isotone map and $f(a) \in \text{p-min}\, g^{-1}(a^{\uparrow})$ for all $a \in A$	g is an isotone map and $f(a) \in \text{p-max}\, g^{-1}(a^{\downarrow})$ for all $a \in A$

following relations hold $(f \circ g \circ f)(a) \approx f(a)$ and $(g \circ f \circ g)(b) \approx g(b)$. Moreover,

- If (f, g) is both left and right Galois connection (resp., adjunction and co-adjunction) then $(g \circ f)(a) \approx a$ and $(f \circ g)(b) \approx b$ for all $a \in A$ and $b \in B$.
- If (f, g) is both a (left or right) Galois connection and a (co-)adjunction then, for all $a_1, a_2 \in A$, $\rho_A(a_1, a_2) = \top$ implies $f(a_1) \approx f(a_2)$ and, for all $b_1, b_2 \in B$, $\rho_B(b_1, b_2) = \top$ implies $g(b_1) \approx g(b_2)$.

Building fuzzy adjunctions on fuzzy posets

Now, we present the main results which lead us to the construction of fuzzy adjunctions between fuzzy posets and fuzzy adjunctions between preordered sets.

Given a mapping f from a fuzzy poset $\langle A, \rho_A \rangle$ to any set B, we will introduce conditions which allow to define a fuzzy ordering on B and a mapping from B to A such that the pair (f, g) forms a fuzzy adjunction.

The problem stated above is addressed from the canonical decomposition of $f \colon \langle A, \rho_A \rangle \to B$ through $A_{\equiv_f}$, the quotient set of A wrt the *kernel relation* $\equiv_f$.

In the following results, we provide the conditions that ensure the definition of a right adjoint for the three mappings, namely $\pi \colon A \to A_{\equiv_f}$ where $\pi(a) = [a]_{\equiv_f}$; the bijective mapping $\varphi \colon A_{\equiv_f} \to f(A)$ defined by $\varphi([a]_{\equiv_f}) = f(a)$ and the inclusion $i \colon f(A) \to B$ that satisfies $f = i \circ \varphi \circ \pi$.

Lemma 3.4: Let $\langle A, \rho_A \rangle$ be a fuzzy poset and let $\sim$ be an equivalence relation on A ($\sim \subseteq A \times A$). Suppose that the following conditions hold

1. there exists $\max[\mathrm{a}]_\sim$, for all $a \in A$.
2. $\rho_A(a_1, a_2) \leq \rho_A(\max[\mathrm{a_1}]_\sim, \max[\mathrm{a_2}]_\sim)$, for all $a_1, a_2 \in A$.

Then, $\rho_{A_\sim} : A_\sim \times A_\sim \to L$ defined by $\rho_{A_\sim}([a_1]_\sim, [a_2]_\sim) = \rho_A(a_1, \max[\mathrm{a}_2]_\sim)$ is a fuzzy ordering on $A_\sim$.

Moreover, the pair $(\pi, \max)$ is a fuzzy adjunction between A and $A_\sim$.

Now, given a bijective mapping $\varphi\colon \langle A, \rho_A\rangle \to B$, we show that φ induces a fuzzy ordering, $\rho_B : B \times B \to \mathbb{L}$ defined as $\rho_B(b, b') = \rho_A(\varphi^{-1}(b), \varphi^{-1}(b'))$ such that φ and φ^{-1} are isotone maps and $(\varphi, \varphi^{-1})\colon A_{\equiv_f} \leftrightharpoons f(A)$ (see Figure 4).

$$\begin{tikzcd}
\langle A, \rho_A\rangle \arrow[rr, "f"] \arrow[d, two heads, shift left, "\pi"] & & B \\
\langle A_{\equiv_f}, \rho_{A_{\equiv_f}}\rangle \arrow[u, dashed, shift left, "\max"] \arrow[r, shift left, "\varphi"] & \langle f(A), \rho_{f(A)}\rangle \arrow[l, dashed, shift left, "\varphi^{-1}"] \arrow[ur, hook, "i"'] &
\end{tikzcd}$$

Figure 4: $(\pi, \max)\colon \mathrm{A} \leftrightharpoons \mathrm{A}_{\equiv_\mathrm{f}}$ and $(\varphi, \varphi^{-1})\colon A_{\equiv_f} \leftrightharpoons f(A)$.

Finally, in order to extend the fuzzy ordering on $f(A)$ to the whole set B, we consider the case of a subset $X \subseteq U$ and a fuzzy order ρ_X on X that can be extended to a fuzzy ordering on U as follows; fix an element $m \in X$ and define $\rho_m\colon U \times U \to L$ as

$$\rho_m(x, y) = \begin{cases} \rho_X(x, y) & \text{if } x, y \in X \\ \rho_X(x, m) & \text{if } x \in X, y \notin X \\ \bot & \text{if } x \notin X, x \neq y \\ \top & \text{if } x \notin X, x = y \end{cases}$$

Then, ρ_m is a fuzzy ordering on U. Moreover, the mapping $j_m\colon \langle X, \rho_X\rangle \to \langle U, \rho_m\rangle$ defined as follows

$$j_m(x) = \begin{cases} x & \text{si } x \in X \\ m & \text{si } x \notin X \end{cases}$$

satisfies that $(i, j_m)\colon \langle X, \rho_X\rangle \leftrightharpoons \langle U, \rho_m\rangle$.

Theorem 3.4: Let $\langle A, \rho_A\rangle$ be a fuzzy poset and consider a mapping $f\colon A \longrightarrow B$. Let $A_{\equiv_f}$ be the quotient set on the kernel relation. Then, there exists a fuzzy ordering ρ_B on B and a mapping $g\colon B \longrightarrow A$ such that $(f, g) : \langle A, \rho_A\rangle \leftrightharpoons \langle B, \rho_B\rangle$ if and only if

1. there exists $\max[a]_{\equiv_f}$ for all $a \in A$.
2. for all $a_1, a_2 \in A$, the following inequality holds:

$$\rho_A(a_1, a_2) \leq \rho_A(\max[a_1]_{\equiv_f}, \max[a_2]_{\equiv_f})$$

In Figure 5, we represent the composition of the three adjunctions which provides a right adjoint of the mapping f.

$$\begin{tikzcd}
\langle A, \rho_A\rangle \arrow[rr, "f"'] \arrow[d, shift left, "\pi"] & & \langle B, \rho_B\rangle \arrow[ll, dashed, bend right, "g=\max\circ\varphi^{-1}\circ j_m"'] \arrow[d, dashed, shift left, "j_m"] \\
\langle A_{\equiv_f}, \rho_{A_{\equiv_f}}\rangle \arrow[u, dashed, shift left, "\max"] \arrow[rr, shift left, "\varphi"] & & \langle f(A), \rho_{f(A)}\rangle \arrow[u, shift left, "i"] \arrow[ll, dashed, shift left, "\varphi^{-1}"]
\end{tikzcd}$$

Figure 5: $(f, g) : \langle A, \rho_A\rangle \leftrightharpoons \langle B, \rho_B\rangle$ such that $g = \max \circ \varphi^{-1} \circ j_m$.

Building fuzzy adjunctions on fuzzy preordered sets

The construction follows the same scheme of that given in Theorem 3.4 as much as possible. But, we need to define a suitable fuzzy version of the p-kernel relation.

Definition 3.4: Let $\mathbb{A} = \langle A, \rho_A\rangle$ be a fuzzy preordered set and consider a mapping $f\colon A \to B$. The *fuzzy p-kernel* relation $\cong_A$ is the transitive closure of the fuzzy union of the relations symmetric kernel $\approx_A$ and kernel $\equiv_f$.

In order to actually build the fuzzy preordering on the codomain B, we make use of a suitable fuzzy preordering between crisp subsets. The idea is to extend the notion of Hoare preorder to a fuzzy setting.

Definition 3.5: Let $\langle A, \rho_A \rangle$ be a fuzzy preordered set, and consider C, D crisp subsets of A. The fuzzy relation $\sqsubseteq_H$ is defined as

$$(C \sqsubseteq_H D) = \bigwedge_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$$

Proposition 3.2: The relation $\sqsubseteq_H$ is a fuzzy preordering in the powerset of A.

It is remarkable that $\sqsubseteq_H$ will be used just on (crisp) subsets $X \subseteq A$ with a particular property; namely, for all $x_1, x_2 \in X$ we have $\rho_A(x_1, x_2) = \top$. A subset is said to be *cyclic* if it satisfies the previous property.

The following lemma states that, for the specific case of this kind of sets, the fuzzy relation $\sqsubseteq_H$ can be very easily computed.

Lemma 3.9: Consider a fuzzy preordered set $\langle A, \rho_A \rangle$, and let X, Y be two crisp cyclic subsets of A. Then, $X \sqsubseteq_H Y = \rho_A(x, y)$ for any $x \in X$ and $y \in Y$.

Notation: Let $\langle A, \rho_A \rangle$ be a fuzzy preordered set and let $X \colon A \to L$ be a fuzzy subset of A. The set of upper bounds of X is defined as follows

$$\mathrm{UB}(X) = \{b \in A \mid X(u) \leq \rho_A(u, b) \text{ for all } u \in A\}$$

The result below actually allows to build a fuzzy preordering relation on B by applying it to the particular case of the sets of p-minima of a fuzzy subset, which turn out to be cyclic (this is just a straightforward consequence of the definition).

Lemma 3.10: Consider a fuzzy preordered set $\mathbb{A} = \langle A, \rho_A \rangle$ together with a mapping $f \colon A \to B$ and a subset $S \subseteq A$ satisfying the following conditions:

1. $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(\text{UB}[a]_{\cong_A} \cap S) \neq \varnothing$, for all $a \in A$.
3. $\rho_A(a_1, a_2) \leq \Big(\text{p-min}(\text{UB}[a_1]_{\cong_A} \cap S) \sqsubseteq_H \text{p-min}(\text{UB}[a_2]_{\cong_A} \cap S)\Big)$, for all $a_1, a_2 \in A$.

Then, for any $a_0 \in A$, the fuzzy relation $\rho_B^{a_0} \colon B \times B \to L$ defined as follows

$$\rho_B^{a_0}(b_1, b_2) = \Big(\text{p-min}(\text{UB}[a_1]_{\cong_A} \cap S) \sqsubseteq_H \text{p-min}(\text{UB}[a_2]_{\cong_A} \cap S)\Big)$$

where $a_i \in f^{-1}(b_i)$ if $f^{-1}(b_i) \neq \varnothing$ and $a_i = a_0$ otherwise, for each $i \in \{1, 2\}$, is a fuzzy preordering on B.

Furthermore, under the same hypotheses, it is possible to define a number of suitable right adjoints $g \colon B \to A$ for f and all of them can be specified as follows:

(C1) If $b \in f(A)$, then $g(b) \in \text{p-min}(\text{UB}[x_b]_{\cong_A} \cap S)$ for some $x_b \in f^{-1}(b)$.

(C2) If $b \notin f(A)$, then $g(b) \in \text{p-min}(\text{UB}[a_0]_{\cong_A} \cap S)$.

We conclude this section stating the theorem which summarizes the necessary and sufficient conditions for the existence of a right adjoint for a mapping between a fuzzy preordering and an unstructured set.

Theorem 3.6: Given a fuzzy preordered set $\mathbb{A} = \langle A, \rho_A \rangle$ together with a mapping $f \colon A \to B$, there exists a fuzzy preordering ρ_B on B and a mapping $g \colon B \to A$ such that $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ if and only if there exists $S \subseteq A$ such that, for all $a, a_1, a_2, \in A$:

1. $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(\text{UB}[a]_{\cong_A} \cap S) \neq \varnothing$
3. $\rho_A(a_1, a_2) \leq \Big(\text{p-min}(\text{UB}[a_1]_{\cong_A} \cap S) \sqsubseteq_H \text{p-min}(\text{UB}[a_2]_{\cong_A} \cap S)\Big)$.

Closure systems on fuzzy preordered sets

The theory of closure systems on preordered sets is used in order to provide a more meaningful framework for the extension to the fuzzy case of previous results.

The notion of closure system on a fuzzy preordered set which we use is a natural extension of the classical closure system on a crisp partial ordered set. In fact, the definition is formulated in the same terms, though we use an alternative characterization that is easier to handle.

Definition 3.8: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set and let $S \subseteq A$ be a crisp subset of A. Then S is said to be a *closure system* if the set p-$\min(a^{\uparrow} \cap S)$ is non-empty, for all $a \in A$.

Other definitions of closure system in a fuzzy setting can be found in the literature. It is remarkable the one given by Belohlavek in [4], where the notions of L_K-closure operator and L_K-closure system on L-ordered sets were introduced, where K is a filter of the residuated lattice L. In that definition, a fuzzy closure system is a fuzzy set, so it is a different approach from ours.

There exists another definition similar in spirit to the one we propose, which was introduced in the framework of the so-called L-ordered sets. In the following result we state an alternative characterization of the notion of closure system based on ideas from [34].

Proposition 3.3: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set. A non-empty subset $S \subseteq A$ is a closure system if and only if for any $a \in A$, there exists $m_a \in S$ such that

1. $\rho_A(a, m_a) = \top$ and
2. $\rho_A(s_1, m_a) \otimes \rho_A(a, s_2) \leq \rho_A(s_1, s_2)$ for any $s_1, s_2 \in S$.

The previous result can be further improved by providing a new characterization which involves just one condition.

Theorem 3.7: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set. A subset S of A is a closure system if and only if for all $a \in A$ there exists $m_a \in S$ satisfying $\rho_A(a, u) = \rho_A(m_a, u)$ for all $u \in S$.

As an easy consequence of this theorem, we obtain a constructive version of the sets p-$\min(a^\uparrow \cap S)$ when S is a closure system.

Corollary 3.2: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set. If $S \subseteq A$ is a closure system then p-$\min(a^\uparrow \cap S) = \{s \in S \mid \rho_A(a, u) = \rho_A(s, u)$ for all $u \in S\}$, for $a \in A$.

It is well-known that closure systems and closure operators in the classical setting are different approaches to the same phenomenon. We focus now on the development of the link between these two notions on fuzzy preordered sets. In order to address this problem, we proceed by proving a number of preliminary results which will pave the way for the characterization.

Definition 3.9: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set. A mapping $c\colon A \to A$ is said to be a *closure operator* if it is isotone, inflationary and satisfies $\rho_A(c(c(a)), c(a)) = \top$ for all $a \in A$.

The following lemma states that the notions of closure system and closure operator keep being interdefinible in the framework of fuzzy preordered sets.

Lemma 3.12: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set.

i) If $S \subseteq A$ is a closure system, then any mapping $c\colon A \to A$ such that $c(a) \in$ p-$\min(a^\uparrow \cap S)$ is a closure operator.

ii) If $c\colon A \to A$ is a closure operator, then $S = \{a \in A : \rho_A(c(a), a) = \top\}$ is a closure system.

In the following, the constructions given in the different items of the previous lemma will be called, respectively, *the closure operator associated to S* (denoted c_S) and *the closure system associated to c* (denoted S_c).

It is well-known that, in (crisp) posets, there exists a one-to-one correspondence between closure operators and closure systems (for every closure operator $c = c_{S_c}$ and for any closure system $S = S_{c_S}$). The relationship between both notions is weaker when the underlying structure is a fuzzy preordered set.

Proposition 3.4: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set.

1. If $c \colon A \to A$ is a closure operator, then

$$\rho_A(c(a), c_{S_c}(a)) = \rho_A(c_{S_c}(a), c(a)) = \top$$

for all $a \in A$.

2. If S is a closure system then $S \subseteq S_{c_S}$ and for all $s_1 \in S_{c_S}$ there exists $s_2 \in S$ such that $\rho_A(s_1, s_2) = \rho_A(s_2, s_1) = \top$.

Now, we define the notion of a closure system compatible wrt an arbitrary fuzzy equivalence relation (a reflexive, symmetric and transitive fuzzy relation) and with the particular case of the so-called *kernel relation*.

Definition 3.10: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set and let $\sim$ be a fuzzy equivalence relation on A.

i) A closure operator $c \colon A \to A$ is said to be *compatible* wrt the relation $\sim$ if $(a_1 \sim a_2) \leq \rho_A(c(a_1), c(a_2))$, for all $a_1, a_2 \in A$.

ii) A closure system $S \subseteq A$ is said to be *compatible* wrt $\sim$ if any closure operator associated to S is compatible wrt $\sim$.

Lemma 3.13: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set and consider a fuzzy equivalence relation $\sim$ on A. Then, a closure system S is compatible with $\sim$ if and only if

$$\rho_A(a,s) \leq \bigwedge_{u \in A} ((a \sim u) \to \rho_A(u,s))$$

for all $s \in S$ and $a \in A$.

Corollary 3.3: Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set, consider a crisp mapping $f\colon A \to B$, and let $\equiv_f$ be the kernel relation associated to f. A closure system $S \subseteq A$ is compatible with the kernel relation if and only if $\rho_A(a,s) = \rho_A(u,s)$ for all $s \in S$ and $a, u \in A$ such that $f(a) = f(u)$.

As one would expect, the mere existence of the adjunction induces a closure system in A which, moreover, is compatible with the kernel relation associated to the mapping $\equiv_f$. This condition is also sufficient as the following theorem shows.

Theorem 3.8: Consider a fuzzy preordered set $\mathbb{A} = \langle A, \rho_A \rangle$ and a mapping $f\colon A \to B$. There exists a fuzzy preordering ρ_B on B and a mapping $g\colon B \to A$ such that (f, g) forms a fuzzy adjunction if and only if there exists $S \subseteq A$ a closure system compatible with the kernel relation $\equiv_f$.

Conclusions and future work

In this thesis, we have provided necessary and sufficient conditions to define suitable (pre)orderings on an unstructured codomain to generate adjunctions, both in crisp case and in a fuzzy setting.

Specifically, given a mapping $f\colon A \to B$ from a (pre)ordered set A into an unstructured set B, we have obtained necessary and sufficient conditions which allow us to define a suitable (pre)ordering relation on B such that there exists a mapping $g\colon B \to A$ such that (f, g) forms an adjunction between (pre)ordered sets.

Whereas the study of the partially ordered case follows more or less the intuition of what should be expected (Theorem 2.6), the description of the conditions on the preordered case is much more involved (Theorem 2.7); only later, when we have considered the use of the $\approx$-closure systems, together with the convenient definition of compatibility wrt the kernel relation $\equiv_f$, in order to rewrite the result in much more concise terms (Theorem 2.9).

In the fuzzy case, we have introduced a characterization of the existence of fuzzy adjunctions in the framework of fuzzy partially ordered sets and fuzzy preordered sets. That is, we have assumed the existence of a mapping $f\colon A \to B$ from a fuzzy poset $\langle A, \rho_A \rangle$ to a set B (not necessarily ordered or fuzzily ordered) and we have characterized when it is possible to define a fuzzy (pre)ordering on B and a mapping $g\colon B \to A$ such that (f, g) forms an adjunction (Theorem 3.4). It is remarkable the fact that the right adjoint is not unique. In fact, there is a number of degrees of freedom in order to define it: just consider the parameterized construction of g that we have given in terms of an element $a_0 \in A$ (in the case of a non-surjective f). Note, however, that our results do not imply that *every* right adjoint should be like that; we simply chose a convenient construction to extent the induced fuzzy ordering on the image of f to the whole set B.

Finally, we have analyzed the different definitions of closure operator and closure system on a fuzzy preordered set, in order to formulate the results concerning the existence of fuzzy preordering relations and adjunctions, in terms of closure systems on fuzzy preordered set (Theorem 3.8).

It is important to underscore that all the results have been stated in terms of adjunctions (isotone Galois connections) but all of them can be straightforwardly modified for using with right Galois connection, left

Galois connection and co-adjunction (in crisp and fuzzy case).

On all these issues we have several proposals to be developed as future work:

- An *L-ordered* set is a triplet $\mathbb{A} = (A, \approx_A, \rho_A)$ where $\approx_A$ is a fuzzy equivalence relation and ρ_A is an L-ordering on A [34]. One interesting line of future work will be the extension of the results in this work to triplet $\mathbb{A} = (A, \approx_A, \rho_A)$ where ρ_A is a fuzzy preorder, i.e. ρ_A is a $\approx_A$-reflexive, $\otimes$-transitive and $\otimes$-$\approx_A$-antisymmetric fuzzy binary relation which is a more general structure.

- When focusing on fuzzy extensions of order relations one can find some interesting developments on the study of both fuzzy partial orders and fuzzy preorders, see [11, 13] for instance. In these works, it is noticed that the versions of antisymmetry and reflexivity commonly used are too strong and, as a consequence, the resulting fuzzy partial orders are very close to the classical case. Accordingly, one interesting line of future work will be the adaptation of our results to these alternative weaker definitions.

- Another source of future work could be the definition of alternative interpretations of the notion of adjunction between multivalued functions (i.e., relations) both in crisp and fuzzy frameworks, with the aim of building a right adjoint for a given multivalued function.

- Concerning potential practical applications of the present work, we will explore the area of *Supervised Learning* and *Classification*, following the ideas developed by Marsala [41, 42]. From previous works by Bělohlávek - De Baets [7] and Kuznetsov [40] who have used FCA techniques to define decision trees, (specifically they used adjunctions to present a method for the construction of such trees) our aim is to propose new discrimination measures, which are prime for the classification of data sets, by building isotone functions (ultimately, adjunctions).

Publications

1. F. García-Pardo, I.P. Cabrera, P. Cordero and M. Ojeda-Aciego. On Galois connections and Soft Computing. IWANN 2013. *Lecture Notes in Computer Science* 7903: pp. $224 - 235$ (2013). CORE B.

2. F. García-Pardo, I.P. Cabrera, P. Cordero, M. Ojeda-Aciego and F.J. Rodríguez. Generating isotone Galois connections on an unstructured codomain. IPMU 2014. *Communications in Computer and Information Science* 443: pp. $91 - 99$ (2014). CORE C.

3. F. García-Pardo, I.P. Cabrera, P. Cordero, M. Ojeda-Aciego and F.J. Rodríguez. On the existence of isotone Galois connections between preorders. ICFCA 2014. *Lecture Notes in Computer Science* 478: pp. $67 - 79$ (2014). CORE C.

4. F. García-Pardo, I.P. Cabrera, P. Cordero, M. Ojeda-Aciego and F.J. Rodríguez. On the definition of suitable orderings to generate adjunctions over an unstructured codomain. *Information Sciences*, 286 : pp. $173 - 187$ (2014). Journal with JCR. Cuartil: Q_1.

5. F. García-Pardo, I.P. Cabrera, P. Cordero and M. Ojeda-Aciego. On the construction of fuzzy Galois connections. ESTYLF 2014. *Proc. of XVII Spanish Conference on Fuzzy Logic and Technology*, pp. $99 - 102$ (2014).

6. F. García-Pardo, I.P. Cabrera, P. Cordero and M. Ojeda-Aciego. On adjunctions between fuzzy preordered sets: necessary conditions. JRS 2014. *Lecture Notes in Artificial Intelligence* 8536: pp. $211 - 221$ (2014).

7. F. García-Pardo, I.P. Cabrera, P. Cordero and M. Ojeda-Aciego. On closure systems and adjunctions between fuzzy preordered sets. ICFCA 2015. *Lecture Notes in Artificial Intelligence* 9113: pp. $114 - 127$ (2015). CORE C.

8. I.P. Cabrera, P. Cordero, F. García-Pardo, and M. Ojeda-Aciego. Constructing right adjoints between fuzzy preordered sets. CAEPIA 2015. *Actas de la XVI Conferencia de la Asociación Española para la Inteligencia Artificial*, pp. $439 - 447$ (2015).

9. F. García-Pardo, I.P. Cabrera, P. Cordero and M. Ojeda-Aciego. On fuzzy preordered sets and monotone Galois connections. IEEE SSCI 2015. *IEEE Symposium Series on Computational Intelligence* 144: pp. $990 - 994$ (2015).

Bibliografía

[1] H. G. Bartel. Formal concept analysis and chemometrics: Chemical composition of ancient egyptian bronze artifacts. *MATCH Commun. Math. Comput. Chem.*, 42:25–38, 2000.

[2] I. Beg. On fuzzy order relations. *J. Nonlinear Science and Applications*, 5:357–378, 2012.

[3] R. Bělohlávek. Fuzzy Galois connections. *Mathematical Logic Quarterly*, 45:497–504, 1999.

[4] R. Bělohlávek. Fuzzy closure operators. *J. of Mathematical Analysis and Applications*, 262:473–489, 2001.

[5] R. Bělohlávek. Lattices of fixed points of fuzzy Galois connections. *Mathematical Logic Quartely*, 47(1):111–116, 2001.

[6] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*. Kluwer, 2002.

[7] R. Bělohlávek, B. De Baets, J. Outrata, and V. Vychodil. Lindig's algorithm for concept lattices over graded attributes. *Lecture Notes in Computer Science*, 4617:156–167, 2007.

[8] R. Bělohlávek and J. Konecny. Concept lattices of isotone vs. antitone galois connections in graded setting: Mutual reducibility revisited. *Information Sciences*, 199:133–137, 2012.

[9] R. Bělohlávek and P. Osička. Triadic fuzzy Galois connections as ordinary connections. In *IEEE Intl Conf on Fuzzy Systems*. IEEE Press, 2012.

[10] G. Birkhoff. *Lattice theory*, volume 25 of *Colloquium Publications*. American Mathematical Society, 1967.

[11] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *Intl J of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8(5):593–610, 2000.

[12] U. Bodenhofer. Representations and constructions of similarity-based fuzzy orderings. *Fuzzy Sets and Systems*, 137(1):113–136, 2003.

[13] U. Bodenhofer, B. De Baets, and J. Fodor. A compendium of fuzzy weak orders: Representations and constructions. *Fuzzy Sets and Systems*, 158(8):811–829, 2007.

[14] F. Börner. Basics of Galois connections. *Lect. Notes in Computer Science*, 5250:38–67, 2008.

[15] G. Castellini, J. Koslowski, and G. Strecker. Categorical closure operators via Galois connections. In *Recent developments of general topology and its applications*, volume 67 of *Mathematical research*, pages 72–79. Akademie Verlag, 1992.

[16] D. Cohen, P. J. P. Creed, and S. Živný. An algebraic theory of complexity for valued constraints: Establishing a Galois connection. *Lect. Notes in Computer Science*, 6907:231–242, 2011.

[17] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.

[18] B. De Baets and H. De Meyer. On the existence and construction of t-transitive closures. *Information Sciences*, 152:167–179, 2003.

[19] K. Denecke, M. Erné, and S. Wismath. *Galois connections and applications*, volume 565 of *Mathematics and its Applications*. Springer, 2004.

[20] J. Denniston, A. Melton, and S. E. Rodabaugh. Formal contexts, formal concept analysis, and Galois connections. In *Electronic Proc. in Theoretical Computer Science*, volume 129, pages 105–120, 2013.

[21] J. Díaz and J. Medina. Multi-adjoint relation equations: Definition, properties and solutions using concept lattices. *Information Sciences*, 253:100–109, 2013.

[22] D. Dikranjan and W. Tholen. *Categorical structure of closure operators*. Kluwer, 1995.

[23] R. P. Dilworth and M. Ward. Residuated lattices. *Transactions of the American Mathematical Society*, 45:335–354, 1939.

[24] M. Erné, J. Koslowski, A. Melton, and G. Strecker. A primer on Galois connections. *Annals of the New York Academy of Sciences*, 704:103–125, 1993.

[25] F. Fages and S. Soliman. Abstract interpretation and types for systems biology. *Theoretical Computer Science*, 403(1):52–70, 2008.

[26] S. Ferré and R. King. BLID: An application of logical information systems to bioinformatics. *Lecture Notes in Computer Science*, 2961:47–54, 2004.

[27] J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer, 1994.

[28] A. Frascella. Fuzzy Galois connections under weak conditions. *Fuzzy Sets and Systems*, 172(1):33–50, 2011.

[29] J. García, I. Mardones-Pérez, M. de Prada-Vicente, and D. Zhang. Fuzzy Galois connections categorically. *Mathematical Logic Quarterly*, 56(2):131–147, 2010.

[30] G. Georgescu and A. Popescu. Non-commutative fuzzy Galois connections. *Soft Computing*, 7(7):458–467, 2003.

[31] G. Gierz et al. *Continuous lattices and domains*. Cambridge University Press, 2003.

[32] J. Goguen. L-fuzzy sets. *J. Mathematical Analysis and Applications*, 18:145–174, 1967.

[33] G. Grätzer. *General Lattice Theory*. Birkhäuser Verlag, 1998.

[34] L. Guo, G.-Q. Zhang, and Q. Li. Fuzzy closure systems on L-ordered sets. *Mathematical Logic Quarterly*, 57(3):281–291, 2011.

[35] J. Järvinen, M. Kondo, and J. Kortelainen. Logics from Galois connections. *Int. J. Approx. Reasoning*, 49(3):595–606, 2008.

[36] D. M. Kan. Adjoint functors. *Transactions of the American Mathematical Society*, 87(2):294–329, 1958.

[37] A. Kerber. Posets and lattices, contexts and concepts. *MATCH Commun. Math. Comput. Chem.*, 54:551–560, 2005.

[38] S. Kerkhoff. A general Galois theory for operations and relations in arbitrary categories. *Algebra Universalis*, 68(3):325–352, 2012.

[39] J. Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.

[40] S. Kuznetsov. Galois connections in data analysis: Contributions from the soviet era and modern russian research. *Lect. Notes in Computer Science*, 3626:196–225, 2005.

[41] C. Marsala, B. Bouchon-Meunier, and A. Ramer. Hierarchical model for discrimination measures. In *Proc. of the IFSA World Congress*, pages 339–343, 1999.

[42] C. Marsala and D. Petturiti. Monotone classification with decision trees. In *Proc of the Conference of the European Society for Fuzzy Logic and Technology*, pages 810–817, 2013.

[43] A. Melton, D. Schmidt, and G. E. Strecker. Galois connections and computer science applications. *Lect. Notes in Computer Science*, 240:299–312, 1986.

[44] M. Moortgat. Multimodal linguistic inference. *Journal of Logic, Language and Information*, 5(3-4):349–385, 1996.

[45] S.-C. Mu and J. N. Oliveira. Programming from Galois connections. *The Journal of Logic and Algebraic Programming*, 81(6):680–704, 2012.

[46] Ø. Ore. Galois connections. *Transactions of the American Mathematical Society*, 55:493–513, 1944.

[47] F. Rioult, J.-F. Boulicaut, B. Crémilleux, and J. Besson. Using transposition for pattern discovery from microarray data. In *Proceedings of the 8th ACM SIGMOD Workshop on Research Issues in Data Mining and Knowledge Discovery*, DMKD '03, pages 73–79, New York, NY, USA, 2003. ACM.

[48] J. Schmidt. Beiträge zur filtertheorie II. *Mathematische Nachrichten*, 10:197–232, 1953.

[49] B. Šešelja and A. Tepavčević. Fuzzy ordering relation and fuzzy poset. *Lecture Notes in Computer Science*, 4815:209–216, 2007.

[50] Y. Shi, M. Nachtegael, D. Ruan, and E. Kerre. Fuzzy adjunctions and fuzzy morphological operations based on implications. *Intl J of Intelligent Systems*, 24(12):1280–1296, 2009.

[51] F. J. Valverde-Albacete and C. Peláez-Moreno. Extending conceptualisation modes for generalised formal concept analysis. *Information Sciences*, 181(10):1888–1909, 2011.

[52] W. Yao and L.-X. Lu. Fuzzy Galois connections on fuzzy posets. *Mathematical Logic Quarterly*, 55:105–112, 2009.

[53] L. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

[54] Q.-Y. Zhang and L. Fan. Continuity in quantitative domains. *Fuzzy Sets and Systems*, 154(1):118–131, 2005.