SHORT COMMUNICATION

Exact BER analysis of $M$-ary orthogonal signaling with MRC over Ricean fading channels

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SUMMARY

A new approach for analyzing the performance of coherent $M$-ary orthogonal signaling with maximal ratio combining over Ricean fading channels is presented here. A new series representation for the bit-error rate is derived, making it possible to compute error rates with guaranteed accuracy. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Orthogonal modulation is very attractive for power-limited wireless communication; thus, analysis of such signal formats in fading channels is of great theoretical and practical importance. The mathematical tractability of coherent $M$-ary orthogonal modulation performance over fading channels is quite restricted, especially when Ricean fading is considered. In a high-power regime, that is, one with high signal-to-noise ratios (SNRs), tight asymptotic approximations for the analysis of the bit-error rate (BER) can be employed [1]. However, limited results are found in the literature [2, 3] for cases when an exact error-rate formulation is applied to a low-power regime (moderate and low SNRs), with an arbitrary number of signals.

Recently, in [3] the analysis of coherent $M$-ary orthogonal signaling over Ricean fading channels with maximal ratio combining (MRC) leads to BER expressions in the form of a single integral

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to be computed numerically with an infinite series of parabolic cylinder functions as integrand. This paper shows that a different approach makes it possible to represent the BER by an infinite series of terms without integrals, thus, without the necessity of resorting to numerical integration, which in turn enables us to strictly bound the associated absolute and relative truncation errors.

2. BER ANALYSIS

The probability density function (pdf) of the SNR for MRC with $L$ independent branches over Ricean fading can be derived from [4]. Assuming line-of-sight components with power $\{x_j^2\}_{j=1}^L$, scatter components with power $2\sigma^2$, energy per symbol $E_s$ and noise power density $N_0$, the following pdf is obtained:

$$p(\gamma_C) = L \frac{1+\bar{K}}{\bar{\gamma}_C} \left( \frac{1+\bar{K}}{\bar{\gamma}_C} \right)^{(L-1)/2} \times \exp \left( -L \left( 1 + \bar{K} \frac{\gamma_C}{\bar{\gamma}_C} + \bar{K} \right) \right) I_{L-1} \left( 2L \sqrt{\bar{K} (1 + \bar{K} \frac{\gamma_C}{\bar{\gamma}_C})} \right)$$

(1)

where $\gamma_C \triangleq \sum_{i=1}^L \tilde{\gamma}_i$ is the SNR per symbol after combining, $\tilde{\gamma}_i$ is the SNR per branch with $\tilde{\gamma}_i \triangleq E[\gamma_i] = (x_i^2 + 2\sigma^2)E_s/N_0$, $\bar{\gamma}_C \triangleq E[\gamma_C]$ and $\bar{K} \triangleq (1/L) \sum_{i=1}^L K_i$ is the average Ricean $K$-factor with $K_i \triangleq x_i^2/2\sigma^2$. Consequently, expression (1) degenerates into the well-known Rayleigh fading pdf when $L = 1$ and $\bar{K} = 0$. After some simple manipulations, the symbol-error rate (SER) of coherent $M$-ary orthogonal modulation in fading channels is expressed as [5]

$$\text{SER} = 1 - \left( \frac{1}{2} \right)^{M-1} \frac{1}{\sqrt{\pi}} E_{\gamma_C} \left[ \int_{q=-\infty}^{\infty} \text{erfc}^{M-1}(-q) \exp(-q - \sqrt{\gamma_C}) dq \right]$$

(2)

where $E_{\gamma_C}[\cdot]$ means expectation over $\gamma_C$ and $\text{erfc}(\cdot)$ is the complementary error function. The BER is simply related to the SER by $\text{BER} = M/(2(M-1)\text{SER})$.

To tackle (2) the following identity is first considered:

$$\text{erfc}^N(x) = \left( \frac{2}{\sqrt{\pi}} \right)^N \int_{t_1=x}^{\infty} \cdots \int_{t_N=x}^{\infty} \exp \left( -\sum_{k=1}^{N} t_k^2 \right) dt_1 \cdots dt_N$$

$$= (v_k = t_k - x) = \left( \frac{2}{\sqrt{\pi}} \right)^N e^{-N x^2} \int_{\mathbb{R}^N_+} \exp(-v^T v - 2x v^T) dv$$

(3)

with $1 \triangleq [1, \ldots, 1]$, $v \in \mathbb{R}^N_+$ and the volume integral extended to the nonnegative orthant $\mathbb{R}^N_+$.

Introducing (3) into (2) and then performing series expansion of the first exponential term within the inner integral and integrating over $q$ with [6, Equation (3.326-2)] yields

$$\text{SER} = 1 - \left( \frac{1}{\sqrt{\pi}} \right)^M E_{\gamma_C} \left[ e^{-\gamma_C} \int_{q=-\infty}^{\infty} \left\{ \int_{\mathbb{R}^{M-1}_+} e^{2q(1^T v + \sqrt{\gamma_C})} e^{-v^T v} dv \right\} e^{-M q^2} dq \right]$$

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EXACT BER ANALYSIS OF M-ARY ORTHOGONAL SIGNALING

\[
1 - \left( \frac{1}{\sqrt{\pi}} \right)^M E \gamma_C \left[ e^{-\gamma_C} \int_{\mathbb{R}^M} \sum_{k=0}^{\infty} \left( \frac{1}{M} \right)^{k+1/2} \frac{4^k \Gamma(k + 1/2)}{\Gamma(2k + 1)} (1^T v + \sqrt{\gamma_C})^{2k} \exp(-v^T v) \, dv \right]^{M-1} \sum_{k=0}^{M^{-k}} \frac{(2k)!}{k! \cdots k_M-1!} \prod_{j=1}^{M-1} \int_0^\infty v^j e^{-v^2} \, dv
\]

where \( \Gamma(\cdot) \) is the gamma function [6].

Considering identity [6, Equation (8.335-1)], Equation (4) is rearranged as

\[
\text{SER} = 1 - \frac{1}{\sqrt{M}} \left( \frac{1}{\sqrt{\pi}} \right)^M M^{-k} E \gamma_C \left[ e^{-\gamma_C} \mathcal{F}_k^+(M, \gamma_C) \right]
\]

where

\[
\mathcal{F}_k^+(M, \gamma_C) \triangleq \int_{R^M} (1^T v + \sqrt{\gamma_C})^{2k} e^{-v^T v} \, dv
\]

Then, after applying the multinomial theorem [7, Equation (24.1.2)], each volume integral \( \mathcal{F}_k^+ \) is again computed with [6, Equation (3.326-2)]

\[
\mathcal{F}_k^+(M, \gamma_C) = \sum_{w=0}^{2k} \gamma_C^{w/2} \sum_{k_1 + \cdots + k_{M-1} = 2k-w} \frac{(2k)!}{k_1! \cdots k_{M-1}!} \prod_{j=1}^{M-1} \int_0^\infty v^j e^{-v^2} \, dv
\]

\[
= (2k)! \sum_{w=0}^{2k} a_{2k-w}(M) \gamma_C^{w/2} / w!
\]

where the coefficients \( a_n(M) \) can be computed recursively as

\[
a_n(M+1) = \sum_{k_1+\cdots+k_{M-1}=n} \prod_{j=1}^{M-1} \frac{\Gamma\left(\frac{k_j + 1}{2}\right)}{2 \cdot k_j!} = \sum_{z=0}^{n} \frac{\Gamma\left(\frac{z + 1}{2}\right)}{2 \cdot z!} a_n(M)
\]

with \( a_n(2) = \frac{\Gamma\left(\frac{n + 1}{2}\right)}{2 \cdot n!} \)

Combining (5) and (7)–(8), it follows that

\[
\text{BER} = \frac{M}{2(M-1)} \left\{ 1 - \sum_{k=0}^{\infty} \sum_{w=0}^{2k} b_{k,w}(M) \mathcal{F}_C \left( z = \frac{w}{2} \right) \right\}
\]

where \( \mathcal{F}_C(z) \triangleq E \gamma_C \left[ e^{-\gamma_C} \gamma_C^z \right] \) and the coefficients

\[
b_{k,w}(M) \triangleq \frac{(2k)! a_{2k-w}(M)}{k! w! M^{k+1/2} \pi^{(M-1)/2}}
\]

are independent of the fading distribution.

In particular, for the pdf given in (1), the function \( \mathcal{F}_{\gamma C}(z) \) can be expressed in terms of tabulated special functions. Specifically, applying Equations (6.643-2) and (9.220-2) of [6] to (1) yields

\[
\mathcal{F}_{\gamma C}(z) = \frac{\Gamma(L + z) \gamma_C^L (1 + \overline{K}) L L^{L+1} e^{-L\overline{K}}}{(L(1 + \overline{K}) + \gamma_C)^{L+z}} \Phi \left( L + z, L; \frac{L^2 \overline{K} (1 + \overline{K})}{L(1 + \overline{K}) + \gamma_C} \right)
\]

(11)

where \( \Phi(\alpha, \beta; \cdot) \) is the confluent hypergeometric function [6, Equation (9.210-1)]. Both special functions \( \Phi(\alpha, \beta; \cdot) \) and \( \Gamma(\cdot) \) required to compute (9) are available in the most commonly used mathematical software packages.

### 3. CONVERGENCE AND TRUNCATION ERROR ANALYSIS

The absolute truncation error associated with the series representation in (9), using \( Q \) terms, is given by

\[
\mathcal{E}(Q) \triangleq \text{BER}(Q) - \text{BER} = \frac{M}{2(M-1)} \left( \frac{1}{\sqrt{\pi}} \right)^{M-1} E_{\gamma C} \left[ e^{-\gamma C} \sum_{k=Q}^{\infty} \frac{M-k}{k!} \gamma^+_k(M, \gamma_C) \right]
\]

(12)

Since (12) contains positive terms, it is only required to bound the volume integral as follows:

\[
\gamma^+_k(M, \gamma_C) < \gamma^+_k(M, \gamma_C) \triangleq \int_{\mathbb{R}^{M-1}} (\mathbf{1}^T \mathbf{v} + \sqrt{\gamma_C})^2 e^{-\mathbf{v}^T \mathbf{v}} d\mathbf{v} = \pi^{(M-1)/2} E_s[(s + \sqrt{\gamma_C})^2]
\]

(13)

where \( s = v_1 + \cdots + v_{M-1} \) is normally distributed as \( \mathcal{N}(0, (M-1)/2) \). Substituting expression (13) into (12) yields

\[
\mathcal{E}(Q) < \mathcal{E}'(Q) \triangleq \frac{M}{2(M-1)} E_{\gamma C} \left[ e^{-\gamma C} E_s \left[ e^{(s + \sqrt{\gamma_C})^2/M} - \sum_{k=0}^{Q-1} \frac{1}{k!} \left( \frac{s + \sqrt{\gamma_C}}{\sqrt{M}} \right)^{2k} \right] \right]
\]

(14)

Note that when \( Q \to \infty \) the right member in (14) vanishes. Since \( \mathcal{E}(Q) \) is nonnegative, this fact provides proof of convergence of the series representation (9). Successively applying Equations (3.323-2), (3.326-2) and (8.335-1) of [6] to (14) leads to the following upper bound for the absolute truncation error of (9):

\[
\mathcal{E}(Q) < \mathcal{E}'(Q) = \frac{M \sqrt{M}}{2(M-1)} \left\{ 1 - \frac{1}{\sqrt{M}} \sum_{k=0}^{Q-1} \frac{1}{k!} \frac{2k!}{M^k} \frac{(M-1)}{4} \frac{1}{(2(k-i))!} \right\}
\]

(15)

It is worth noting that, for any \( Q \), the value of the truncated series \( \text{BER}(Q) \) is strictly greater than the value of \( \text{BER} \). When \( Q \) is high enough, \( \mathcal{E}'(Q) < \text{BER}(Q) \), thus, the relative truncation error is upper bounded as follows:

\[
\delta(Q) \triangleq \frac{\mathcal{E}'(Q)}{\text{BER}(Q) - \mathcal{E}'(Q)} \leq \frac{\mathcal{E}(Q)}{\text{BER}(Q) - \mathcal{E}'(Q)}
\]

(16)
4. NUMERICAL RESULTS AND CONCLUSIONS

Figure 1 shows the average BER as a function of the average SNR per bit per branch \( \bar{\gamma}_{bb} \equiv \bar{\gamma}_C/L/\log_2 M \) when all the branches are identical. The number of terms \( Q \) is dynamically increased until the relative truncation error is guaranteed to be \( \delta \leq 1\% \), which is not straightforward with the analysis in [3], where numerical errors associated with the integral [3, Equation (11)] are not included in the error bound [3, Equation (16)]. Therefore, two benefits are achieved with the analysis presented in this paper: firstly, a new series representation without any integral for the BER of coherent \( M \)-ary signaling over generalized fading channels is derived; second, expressions that guarantee a specific target accuracy are provided.

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