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SOME OPEN PROBLEMS ON A CLASS OF INTEGRAL OPERATORS ON SPACES OF ANALYTIC FUNCTIONS

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If g is a fixed analytic function in the unit disc $\mathbb{D}$, the linear operator T_g is defined for analytic functions f in $\mathbb{D}$ by

$$T_g f(z) = \int_0^z f(t)g'(t)dt.$$

These operators appear in a natural way in many problems in complex analysis and they are closely related to a number of important operators including the integral operator and the Cesáro operator. The main results about the operators T_g are the characterization of the symbols g for which T_g acts as a bounded or compact operator on Hardy and weighted Bergman spaces, see [2], [3] and [1].

The next natural question would then be to determine the spectra of such operators. The simplest observation is that T_g cannot have eigenvalues. In particular, it follows that whenever T_g is compact on some space of analytic functions, we have $\sigma(T_g) = \{0\}$ (see Section 8 of [1]). Another simple situation is when g is bounded, Proposition 8.1 of [1] asserts that in such a case $\sigma(T_g|H^2) = \sigma(T_g|L_a^{2,\alpha}) = \{0\}$.

Problem 1. *For $g \in BMOA$ describe the spectrum of the operator T_g on H^2.*

This problem comes from [1] where the following more concrete question is raised:

Is it true that for every $g \in BMOA$ we have

$$\sigma(T_g|H^2) = \{0\} \cup \{\lambda \neq 0 : \ e^{g/\lambda} \notin H^2\}\,?$$

We remark that the answer is affirmative if either:

- $g \in VMOA$, because then T_g is compact and $e^{g/\lambda} \in H^2$, for all λ or
- if g' is of the form

$$g' = r + h'$$

  where r is a rational function with poles on the unit circle and $h \in VMOA$, see [4, Theorem 5.2].

Here is a set of three, more specialized, questions about these operators. Let $g \in H^\infty$. As mentioned above, the operator T_g acting on H^2 is a bounded quasinilpotent operator, i. e., $\sigma(T_g) = 0$.

Problem 2. *Is it true that every nontrivial invariant subspace of T_g has the form $z^N H^2$ for some positive integer N?*

Problem 3. *If $g \notin VMOA$ then T_g is not a compact operator. Is T_g^n compact for some positive integer n?*

The general version of Question 2 is, of course the following.

Problem 4. *Given a positive integer n, find a characterization of the symbols g with the property that T_g^n is compact on H^2 (or on $L_a^{2,\alpha}$, $\alpha > -1$).*

References

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